Algorithms for isogeny graphs

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Cryptographic motivation

We need an abelian variety of small dimension (i.e. 1,2) defined over $\mathbb{F}_q$ s.t. $\#A(\mathbb{F}_q)$ is divisible by a large prime number

For pairing based cryptography, use the complex multiplication method to generate curves with prescribed number of points.

$\longrightarrow$ needs precomputing the class polynomials

- Let J be a (simple) abelian surface over $\mathbb{C}$.
- $\mathrm{End}(J)$ is an order of a (primitive) quartic CM field K (totally imaginary quadratic extension of a totally real number field).
- The class polynomials $H_1, H_2, H_3 \in \mathbb{Q}[X]$ parametrize the invariants of all abelian varieties $A/\mathbb{C}$ with $\mathrm{End}(A) \simeq \mathcal{O}_K$.

Assume p is a "good" prime

$$H_i(X) = \prod_{\mathrm{End}(A) \simeq \mathcal{O}_K} (X - j_i(A))$$

$\#J(\mathbb{F}_p) = N_{K/\mathbb{Q}}(\pi - 1)$, where π is the Frobenius endomorphism.

The CRT method for class polynomial computation

Eisenträger, Freeman, Lauter, Bröker, Gruenewald, Robert :

- Select a "good" prime p.
- For each abelian surface J in the p^3 isomorphism classes
  - Check if J is in the right isogeny class.
  - Check if $\mathrm{End}(J) \simeq \mathcal{O}_K$.
- Reconstruct H_i mod p from jacobians with maximal endomorphism ring

Compute class polynomials modulo small "good" primes and use the CRT to reconstruct H_1, H_2, H_3.

Computing all abelian varieties with maximal order

Eisenträger, Freeman, Lauter, Bröker, Gruenewald, Robert :

- Select a “good” prime p.
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  - Check if J is in the right isogeny class.
  - Check if $\mathrm{End}(J) \simeq \mathcal{O}_K$.
  - Generate jacobians with CM by $\mathcal{O}_K$ by computing horizontal isogenies* from J.
- Reconstruct H_i mod p from jacobians with maximal endomorphism ring

*An isogeny $I : J_1 \to J_2$ is horizontal iff $\mathrm{End}\, J_1 \simeq \mathrm{End}\, J_2$.

Pairings and endomorphism rings

I.-Joux 2010 : algorithms for horizontal isogeny and endomorphism ring computation in genus 1 by using the Tate pairing

F. Morain : *"je suis sûr qu'il y a quelque chose à dire sur les matrices du Frobenius. De toute façon, tout est dans le Frobenius!"*

meaning

"It's all about the Frobenius!"

Claim : *Indeed, but from a computational point of view, using pairings is faster in many cases*.

$\mathrm{End}(J) \otimes \mathbb{Z}_\ell \to \mathrm{End}_{\mathbb{F}_q}(T_\ell(J))$ bijectively

The endomorphism ring of an ordinary jacobian

Let K be a quartic CM field and assume that $K = Q(\eta)$ with

$$\eta = i\sqrt{a + b\frac{-1+\sqrt{d}}{2}} \text{ for } d \equiv 1 \bmod 1$$
$$\eta = i\sqrt{a + b\sqrt{d}} \text{ for } d \equiv 2, 3 \bmod 4$$

Assume real multiplication $\mathcal{O}_{K_0}$ has class number 1.

Let J be a jacobian of a genus 2 curve defined over $\mathbb{F}_q$.

J is ordinary, i.e. $\mathrm{End}(J)$ is an order of K.

$$\mathbb{Z}[\pi, \bar{\pi}] \subset \mathrm{End}(J) \subset \mathcal{O}_K$$

Computing endomorphism rings

Eisenträger and Lauter's algorithm (2005), Freeman-Lauter (2008)

Idea: If $\alpha : J \to J$ is an endomorphism, then $\frac{\alpha}{n}$ is an endomorphism iff $J[n] \subset \mathrm{Ker}\ \alpha$.

Check if an order $\mathcal{O}$ is contained in $\mathrm{End}(J)$:

- Write down a basis for the order $\mathcal{O}$: $\gamma_i = \frac{\alpha_i}{n_i}$, with $\alpha_i \in \mathbb{Z}[\pi]$.
- Check if $\gamma_i \in \mathrm{End}(J)$ by checking if α_i is zero on $J[n_i]$.

Since $n_i | [\mathcal{O}_K : \mathbb{Z}[\pi, \bar{\pi}]]$ we end up working over large extension fields!

Just to give an idea...

The smallest extension field $\mathbb{F}_{q^r}$ s.t. $J[\ell] \subset J(\mathbb{F}_{q^r})$ has degree r at most ℓ^4.

$$\text{If } J[\ell^2] \not\subseteq J(\mathbb{F}_{q^r}), \text{ then } J[\ell^2] \subseteq J(\mathbb{F}_{q^{r\ell}})$$

$$J[\ell^3] \subseteq J[\mathbb{F}_{q^{r\ell^2}}]$$

$$\ldots$$

Bottleneck: group structure computation $\implies \ell$ is small

Computing the endomorphism ring

- For small ℓ, use Eisenträger-Lauter
- If ℓ is larger, use Bisson's algorithm (2012)
  - *smooth* relations in the class group of the order $\mathcal{O}$
  - corresponding *smooth* horizontal isogeny chains

$$O((exp\sqrt{\log q \log\log q}))^{2\sqrt{3}+o(1)}$$

under GRH and other heuristic assumptions

Notations

Let $\theta \in \mathcal{O}$. We define

$$v_{\ell,\mathcal{O}}(\theta) := max_{a\in\mathbb{Z}}\{m|\theta + a \in \ell^m\mathcal{O}\}$$

How do we compute this?

Consider a $\mathbb{Z}$-basis $1, \delta, \gamma, \eta$ for $\mathcal{O}$:

Write $\theta = a_1 + a_2\delta + a_3\gamma + a_4\eta$. Then

$$v_{\ell,\mathcal{O}}(\theta) := v_\ell(\text{gcd}(a_2, a_3, a_4)).$$

Checking locally maximal orders at ℓ

In general, $v_{\ell,\mathcal{O}}(\theta) \leq v_{\ell,\mathcal{O}_K}(\theta)$

Take $O_{K_0} = [1, \omega]$ and $\eta = i\sqrt{a + b\omega}$, with $(b, \ell) = 1$. Then $\theta = a_1 + a_2\omega + (a_3 + a_4\omega)\eta$, $a_i \in \mathbb{Z}$.

Lemma $*$

Let $\mathcal{O}$ be an order such that $\theta \in \mathcal{O}$ and $[\mathcal{O}_K : \mathcal{O}]$ is divisible by a power of ℓ. If $\max(v_\ell(\frac{a_3 - a_4}{\ell}), v_\ell(\frac{\ell a_3 - a_4}{\ell^2})) < \min(v_\ell(a_3), v_\ell(a_4))$ then $v_{\ell,\mathcal{O}}(\theta) < v_{\ell,\mathcal{O}_K}(\theta)$.

Let $v_\ell(\pi) = v_{\ell,\mathrm{End}(J)}(\pi)$.

A simple criterion: check if $v_\ell(\pi) = v_{\ell,\mathcal{O}_K}(\pi)$.

Checking locally maximal orders at ℓ

How do we compute $v_\ell(\pi)$?

Proposition

$v_\ell(\pi)$ is the largest integer m such that the Frobenius action on $T_\ell(J)$ is a multiple of the identity up to precision m.

The matrix of the Frobenius is of the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \bmod \ell^k, k \leq m$$

We could compute the action of the Frobenius on $J[\ell]$, $J[\ell^2]$ ...

This means working over large extension fields very quickly, so NO!

How do we compute $v_\ell(\pi)$?

2006 Schmoyer : bring pairings into play!

Let A be an abelian variety defined over a field K.
$A[m]$ is the m-torsion and $\hat{A}[m] \simeq \mathrm{Hom}(A[m], \mu_m)$.

Weil pairing

$$e_m : A[m] \times \hat{A}[m] \to \mu_m$$

is a bilinear, non-degenerate map.

If A is a principally polarized variety

$$\begin{aligned} e_m : A[m] \times A[m] &\to \mu_m \\ (P, Q) &\to e_m(P, Q). \end{aligned}$$

The Tate pairing

We denote by $G_K = \mathrm{Gal}(\bar{K}/K)$ the Galois group.

Consider $0 \to A[m] \to A(\bar{K}) \stackrel{m\cdot}{\to} A(\bar{K}) \to 0$.

Take Galois cohomology and get connecting morphism

$$\begin{array}{rcl} \delta : A(K)/mA(K) = H^0(G_K, A)/mH^0(G_K, A) & \to & H^1(G_K, A[m]) \\ P & \to & F_P, \end{array}$$

where we take $\bar{P}$ such that $m\bar{P} = P$ and define

$$\begin{array}{rcl} F_P(\sigma) : G_K & \to & A(\bar{K})[m] \\ \sigma & \to & \sigma \cdot \bar{P} - \bar{P}. \end{array}$$

The Tate pairing

We get the map

$$\begin{array}{rcl} A(K)/mA(K) \times \hat{A}[m](K) & \to & H^1(G_K, \mu_m) \\ (P, Q) & \to & [\sigma \to e_m(F_P(\sigma), Q)] \end{array}$$

bilinear, non-degenerate

The Tate pairing

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bilinear, non-degenerate

The Tate pairing

For a principally polarized abelian variety over a finite field $\mathbb{F}_q$ s.t. $\mu_m \subset \mathbb{F}_q$

$$H^1(G_{\mathbb{F}_q}, \mu_m) \simeq H^1(Gal(\mathbb{F}_{q^m}/\mathbb{F}_q), \mu_m) \simeq \mu_m$$

We take $\bar{P} \in A(\bar{F}_q)$ such that $m\bar{P} = P$ and define

The Tate pairing

$$\begin{array}{rcl} A(\mathbb{F}_q)/mA(\mathbb{F}_q) \times A[m](\mathbb{F}_q) & \rightarrow & \mu_m \\ (P, Q) & \rightarrow & e_m(\pi(\bar{P}) - \bar{P}, Q) \end{array}$$

Pairings on kernels

Assume there is $n \geq 1$ is s.t. $J[\ell^n] \subseteq J[\mathbb{F}_q]$ and $J[\ell^{n+1}] \not\subseteq J[\mathbb{F}_q]$, $\ell > 2$ prime (or $\pi - 1$ is divisible exactly by ℓ^n)

Let $\mathcal{W}$ be the set of subgroups G of rank 2 in $J[\ell^n]$ which are maximal isotropic with respect to the Weil pairing.

$$k_{\ell,J} := \max_{G \in \mathcal{W}} \{k | \exists P, Q \in G \text{ s.t. } T_{\ell^n}(P, Q) \in \mu_{\ell^k} \backslash \mu_{\ell^{k-1}}\}$$

One pairing, two formulae

$$A(\mathbb{F}_q)/\ell^n A(\mathbb{F}_q) \times A[\ell^n](\mathbb{F}_q) \to \mu_{\ell^n}$$

Tate

$$(P, Q) \to e_{\ell^n}(\pi(\bar{P}) - \bar{P}, Q)$$

with $\ell^n \bar{P} = P$ and $\bar{P} \notin J(\mathbb{F}_q)$

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Lichtenbaum

$$(P, Q) \to (f_{P,\ell^n}(Q+R)/f_{P,\ell^n}(R))^{\frac{q-1}{\ell^n}}$$

with f_{P,ℓ^n} s.t. $\mathrm{div}(f_{P,\ell^n}) \sim \ell^n(P)$

$\Leftarrow$ compute in $O(n \log \ell + \log q)$ op. in $\mathbb{F}_q$.

One pairing, two formulae

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compute the Frobenius action up to precision $\geq n$. $\Leftarrow$ compute in $O(n \log \ell + \log q)$ op. in $\mathbb{F}_q$.

Computing $v_\ell(\pi)$

Theorem

Suppose $\pi - 1$ is exactly divisible by ℓ^n and $0 < v_{\ell,\mathcal{O}_K}(\pi) < 2n$. Then $v_\ell(\pi) = 2n - k_{\ell,J}$.

Proof: Galois cohomology+linear algebra

Corollary

If $0 < v_{\ell,\mathcal{O}_K}(\pi) < 2n$ and under the conditions of Lemma $*$, then $\mathrm{End}(J)$ is a locally maximal order at ℓ iff $k_{\ell,J} = 2n - v_{\ell,\mathcal{O}_K}(\pi)$.

Computational issues

We need to get $k_{\ell,J} = \max_{G \in \mathcal{W}} \{k | T_{\ell^n} : G \times G \to \mu_{\ell^k} \text{ surjective}\}$.

There are $O(\ell^3)$ subgroups in $\mathcal{W}$!

In practice, compute a symplectic basis $\{Q_1, Q_{-1}, Q_2, Q_{-2}\}$.

Get $k_{\ell,J} = \max_{j \neq -i} \{k | T_{\ell^n}(Q_i, Q_j) \text{ is a } \ell^k\text{-th primitive root of unity}\}$

Algorithm

- If the $J[\ell]$ is not defined over $\mathbb{F}_q$, switch to $\mathbb{F}_{q^r}$, $r \leq \ell^4 - 1$.
- Compute largest integer n s.t. $J[\ell^n] \subset J(\mathbb{F}_{q^r})$.
- Compute a symplectic basis $\{Q_1, Q_{-1}, Q_2, Q_{-2}\}$.
- Compute $k_{\ell,J} = \max_{i \neq -j}\{k | T_{\ell^n}(Q_i, Q_j) \text{ is a } \ell^k\text{-th primitive root of unity}\}$
- If $v_{\ell,\mathcal{O}_K}(\pi^r) = 2n - k_{\ell,J}$ return true.

Complexity analysis

Denote by $\mathbb{F}_{q^r}$ the smallest extension field such that $J[\ell] \subset J[\mathbb{F}_{q^r}]$.

Let $n \geq 1$ be the largest integer such that $J[\ell^n] \subset J(\mathbb{F}_q)$ and $u = v_\ell([\mathcal{O}_K : \mathbb{Z}[\pi, \bar{\pi}]])$.

Let $M(r)$ is the cost of a multiplication in F_{q^r}.

Freeman-Lauter	This work
$O((r\ell^{u-n} + \ell^{2u})M(r\ell^{u-n}) \log q)$ (worst case)	$O(M(r)(r \log q + \ell^{2n} + n \log \ell))$

Heuristically, if u is large, we would expect $u > n$.

Example

Consider $y^2 = 27x^6 + 869x^5 + 364x^4 + 407x^3 + 518x^2 + 47x + 806$ over $\mathbb{F}_{1009}$.

The index is $[\mathcal{O}_K : \mathbb{Z}[\pi, \bar{\pi}]] = 3^4$. The 3-torsion is defined over $\mathbb{F}_{1009^2}$.

$\pi^2 = 8626 - 234\frac{1+\sqrt{109}}{2} + (-33 + 27\frac{1+\sqrt{109}}{2})\sqrt{702 - 13\frac{1+\sqrt{109}}{2}} \implies v_{\ell,\mathcal{O}_K}(\pi^2) = 1$.

It took less then 2 seconds on a AMD Phenom II X2 B55 (3 GHz) to compute $k_{\ell,J} = 1$ and decide that $\mathrm{End}(J)$ is locally maximal at ℓ.

The Freeman-Lauter algorithm runs over $\mathbb{F}_{1009^6}$ and returns the same result in 60 sec.

The CRT method for computing class polynomials

- Select a "good" prime p.
- For each abelian surface J in the p^3 isomorphism classes
  - Check if J is in the right isogeny class.
  - Check if $\mathrm{End}(J) \simeq \mathcal{O}_K$.
  - Generate jacobians with CM by $\mathcal{O}_K$ by computing horizontal isogenies from J.
- Reconstruct H_i mod p from jacobians with maximal endomorphism ring

Compute class polynomials modulo small "good" primes and use the CRT to reconstruct H_1, H_2, H_3.

Computing horizontal isogenies

An ℓ-isogeny is an isogeny whose kernel is a subgroup of $J[\ell]$ maximal isotropic with respect to the Weil pairing.

An ℓ-isogeny $I : J_1 \to J_2$ is horizontal iff $\mathrm{End}\, J_1 \simeq \mathrm{End}\, J_2$.

Given by the action of the Shimura class group

$$\{(\mathfrak{a}, \alpha) | \mathfrak{a} \text{ is a fractional } \mathcal{O}_K\text{-ideal with } \mathfrak{a}\bar{\mathfrak{a}} = (\alpha) \text{ with } \alpha \in K_0 \text{ totally positive}\}/K^*$$

Let ℓ coprime to discriminant of $\mathbb{Z}[\pi, \bar{\pi}]$. Then the kernel of $I_{\mathfrak{a}}$ is a subgroup invariated by π.

$$O(M(r)(r \log q + \ell^{2n}))$$

Non-degenerate pairing on kernel

Let J be a jacobian whose endomorphism ring is locally maximal at ℓ.

Assume $\pi - 1$ is exactly divisible by ℓ^n and let G be a subgroup in $\mathcal{W}$.

The Tate pairing is non-degenerate on $G \times G$ if

$$T_{\ell^n} : G \times G \to \mu_{\ell^{k_{\ell,J}}}$$

is surjective. We say it is degenerate otherwise.

Let G_1 be a maximal isotropic subgroup of $J[\ell]$.
Consider $G \in \mathcal{W}$ such that $\ell^{n-1} G = G_1$.

Theorem

- If the isogeny of kernel G_1 is horizontal, then the Tate pairing is degenerate on $G \times G$.
- Under the conditions from Lemma $*$, if the Tate pairing is degenerate on $G \times G$, then the isogeny is horizontal.

$$O(M(r)(r \log q + \ell^{2n} + n \log \ell))$$

An example

We consider the jacobian of the hyperelliptic curve

$$y^2 = 5x^5 + 4x^4 + 98x^2 + 7x + 2, \text{ over } \mathbb{F}_{127}.$$

$\mathrm{End(J)}$ is maximal at 5 and $[\mathrm{End}J : \mathbb{Z}[\pi, \bar{\pi}]] = 50$.
The decomposition $(5) = \mathfrak{a}\bar{\mathfrak{a}}$ in $\mathcal{O}_K$ gives two horizontal isogenies.
The 5-torsion is defined over $\mathbb{F}_{127}(t) := \mathbb{F}_{127^8}$.
With MAGMA, we computed the Mumford coordinates of the generators of kernels:

$$(x^2 + (74t^7 + 25t^6 + 6t^5 + 110t^4 + 96t^3 + 75t^2 + 29t + 20)x + 39t^7 + 62t^6 + 77t^5 + 47t^4$$
$$+9t^3 + 62t^2 + 97t + 15, (116t^7 + 61t^6 + 13t^5 + 38t^4 + 70t^3 + 109t^2 + 62t + 71)x + 98t^7 + 77t^6 + 17t^5$$
$$+76t^4 + 81t^3 + 5t^2 + 36t + 33), (x^2 + (66t^7 + 89t^6 + 50t^5 + 124t^4 + 91t^3 + 102t^2 + 100t + 52)x + 119t^7$$
$$+14t^6 + 126t^5 + 42t^4 + 42t^3 + 85t^2 + 12t + 77, (92t^7 + 90t^6 + 94t^5 + 57t^4 + 59t^3 + 24t^2 + 72t$$
$$+11)x + 103t^7 + 16t^6 + 7t^5 + 111t^4 + 95t^3 + 79t^2 + 45t + 34)$$

Kernels with non-degenerate pairing

There are $\ell^3 + \ell^2 + \ell + 1$ ℓ-isogenies. Experimentally, we observed:

ℓ	$\#\ell$-Isogenies	$\#$Kernels with deg. pairing
3	40	4, 7, 8
5	156	6, 8, 12
7	400	8, 14, 16
11	1464	12, 22, 24

It seems that at most $O(\ell)$ subgroups in $\mathcal{W}$ have degenerate Tate pairing.

Future work

- In genus 1, the ℓ-adic valuation of the Frobenius fully characterizes the endomorphism ring.

  I.-Joux, Pairing the volcano, *Math. Comp.*
  `http://arxiv.org/abs/1110.3602`

- In genus 2, we need a stronger invariant. Work in progress with Emmanuel Thomé.

  I., Pairing-based algorithms for jacobians of genus 2 curves with maximal endomorphism ring,
  `http://fr.arxiv.org/abs/1204.0222`