Divide and conquer roadmap : deciding connectivity for real algebraic sets

Marie-Françoise Roy

IRMAR/Université de Rennes 1

Email: marie-francoise.roy@univ-rennes1.fr

1-Introduction

real algebraic set defined in $\mathbb{R}^k$ by equations of degree d
number of connected components $O(d)^k$
(polynomial in the degree, and singly exponential in the number of variables, sharp)
famous result by Oleinik, Petrowski, Thom and Milnor

Important problems

- deciding connectivity : designing (efficient) algorithms for deciding whether two points belong to the same connected component of an algebraic set

- counting the number of connected components of an algebraic set

Related to robot motion planing, and also to computing the topology, such as determining the Betti numbers.

Cylindrical algebraic decomposition

Schwartz-Sharir, first algorithm for solving this problem, based on **Collins** cylindrical algebraic decomposition
-doubly exponential complexity
-use of resultant, subresultants
-eliminating one variable produces polynomials of degree d^2 in $k-1$ variables
-possible to control adjacencies between cells, if all polynomials monic with respect to the elimination variable (through a linear change of variable)

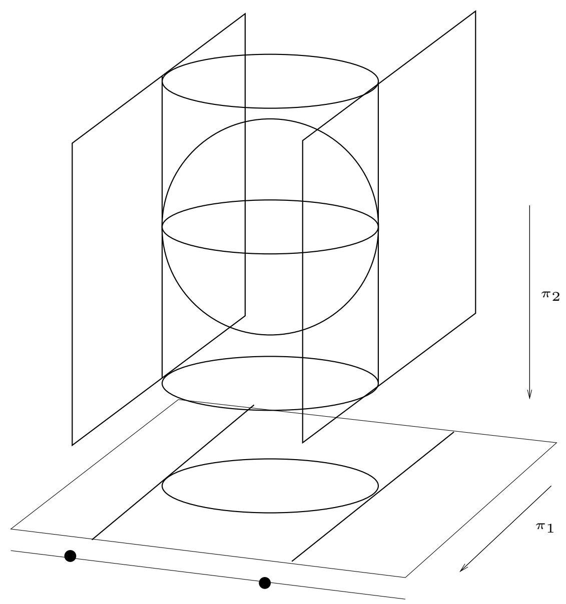

Figure 1. A cylindrical decomposition adapted to the sphere in $\mathbb{R}^3$

Goals of the talk: avoid double exponential complexity

2- describe the classical roadmap construction

3- give motivations to do better

4- describe the baby step - giant step results of **Basu, R., Safey, Schost**

5- discuss divide and conquer current research by **Basu, R.** giving divide and conquer roadmaps

2- Classical roadmaps

Definition of roadmap

what is meant by a roadmap ?

Definition 1. $S \subset R^k$, *semi-algebraic set,* $\mathcal{M} \subset S$ *finite set of points* *roadmap for* $(S, \mathcal{M})$ *: semi-algebraic set* $\mathrm{RM}(S, \mathcal{M})$ *of dimension at most one contained in* S *satisfying:*

1. RM_1 *For every connected component* C *of* S, $C \cap \mathrm{RM}(S, \mathcal{M})$ *is connected.*
2. RM_2 *For every* $x \in R$ *and for every connected component* D *of* S_x, $D \cap \mathrm{RM}(S, \mathcal{M}) \neq \emptyset$,
   ($S_x = S \cap \pi_1^{-1}(x)$ *for* $x \in R$, *and* $\pi_1 \colon R^k \to R$ *the projection map onto the first coordinate)*

Outline of the classical roadmap algorithm

bounded non singular (generic) hypersurface $\mathrm{Zer}(Q, R^k)$
geometric ideas due to **Canny**
description based on **Basu, Pollack, R.**

key ingredient ; construction of a finite set of points intersecting every connected component of $\mathrm{Zer}(Q, R^k)$: X_1-critical points on $\mathrm{Zer}(Q, R^k)$

for more general situations, X_1-pseudo-critical points=limits of the X_1-critical points of a bounded nonsingular algebraic hypersurface defined by a particular infinitesimal deformation of the polynomial Q

projections on the X_1-axis : X_1 – pseudo-critical values

key connectivity property : connectivity controlled by pseudo-critical values

Proposition 2. *Let* $\mathrm{Zer}(Q, R^k)$ *be a bounded algebraic set and* S *a connected component of* $\mathrm{Zer}(Q, R^k)_{[a,b]}$. *If* $[a, b] \setminus \{v\}$ *contains no* X_1*-pseudo-critical value on* $\mathrm{Zer}(Q, R^k)$, *then* S_v *is semi-algebraically connected.*

for classical roadmap: X_2-pseudo-critical points parametrized by X_1

construct the "silhouette" : set of X_2-pseudo-critical points on $\mathrm{Zer}(Q, R^k)$ parametrized by X_1, obtained by following, as x varies on the X_1-axis, the X_2-pseudo-critical points on $\mathrm{Zer}(Q, R^k)_x$

defines curves and endpoints on $\mathrm{Zer}(Q, R^k)$

include for every $x \in R$ the X_2-pseudo-critical points of $\mathrm{Zer}(Q, R^k)_x$, so meet every semi-algebraically connected component of $\mathrm{Zer}(Q, R^k)_x$.

set of curves satisfy RM_2

however, RM_1 might not be satisfied

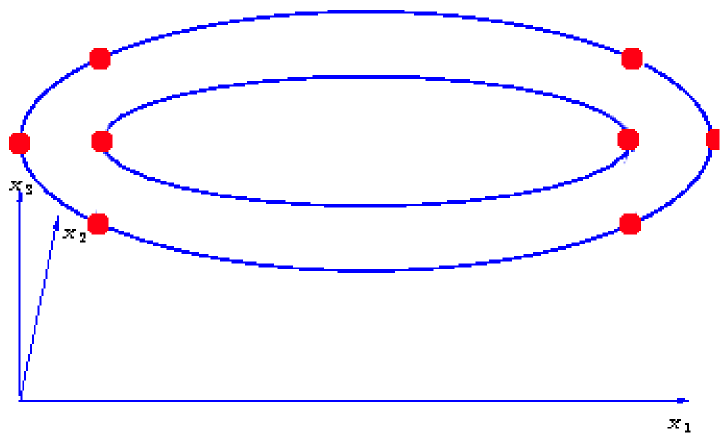

Figure 2. X_2-(pseudo-)critical points on a torus in $\mathbb{R}^3$

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to ensure property RM_1, needed to add more curves to the roadmap

distinguished values $\mathcal{D}$: union of the X_1-pseudo-critical values, and the first coordinates of the endpoints of the curves

distinguished hyperplane : hyperplane defined by $X_1 = v$, where v is a distinguished value

$$\text{distinguished values} : v_1 < \ldots < v_N$$

above each interval (v_i, v_{i+1}) collection of curves $\mathcal{C}_i$ meeting every connected component of $\mathrm{Zer}(Q, R^k)_v$ for every $v \in (v_i, v_{i+1})$

above each distinguished value v_i, a set of distinguished points $\mathcal{M}_i$

each curve in $\mathcal{C}_i$ has an endpoint in $\mathcal{M}_i$ and another in $\mathcal{M}_{i+1}$

the union of the $\mathcal{M}_i$ contains $\mathcal{M}$

$$\mathcal{C} \text{ the union of the } \mathcal{C}_i$$

Definition 3. S, S^0, S^1 *semi-algebraic sets with* $S^0 \subset S$, $S^1 \subset S$
(S, S^0, S^1) *has good connectivity property if for every connected component* C *of* S,
$C \cap (S^0 \cup S^1)$ *is connected.*

key connectivity result proved in **Basu, Pollack, R.**

Proposition 4. $(\mathrm{Zer}(Q, R^k), \mathcal{C}, \mathrm{Zer}(Q, R^k)_{\mathcal{D}})$ *has good connectivity property*

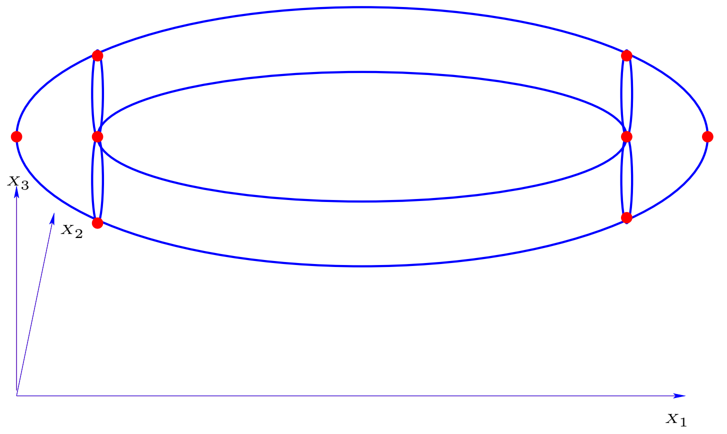

Figure 3. The roadmap of the torus

in order to construct a roadmap of $\mathrm{Zer}(Q, R^k)$, repeat the construction in each distinguished hyperplane H_i defined by $X_1 = v_i$ with input $Q(v_i, X_2, ..., X_k)$ and the distinguished points in $\mathcal{M}_i$ by making recursive calls to the algorithm

proposition proved in **Basu, Pollack, R.**.

Proposition 5. $\mathrm{RM}(\mathrm{Zer}(Q, R^k), \mathcal{M})$ *roadmap for* $(\mathrm{Zer}(Q, R^k), \mathcal{M})$.

at each recursive call, hypersurface, in a smaller dimensional ambiant space

Complexity analysis

$d^{O(k^2)}$ number of recursive calls in the roadmap algorithm :

depth of the recursion k

$$d^{O(k^2)} = \underbrace{d^{O(k)} \times \cdots \times d^{O(k)}}_{k\,\text{times}}$$

in the recursive calls, computation in an algebraic extension of the base field since the distinguished values are algebraic numbers, necessary to analyze carefully the complexity of each arithmetic operation over this extension in terms of the number of operations in the base ring

Summary

classical roadmap algorithms based on **Canny**'s construction proceeds by

-first construct the "silhouette", consisting of curves in the X_1-direction intersecting each connected component of the fiber

-then make recursive calls to the same algorithm at hyperplane sections of $\mathrm{Zer}(Q, R^k)$ where the description of the silhouette changes to ensure connectivity inside every connected component

the number of hyperplane sections is $O(d)^k$

the dimension of the ambient space drops by 1 at each recursive call

the degree with the remaining variables remains d

complexity $d^{O(k^2)}$

Discussion on classical roadmap

Canny (followed by others : **Gournay Risler, Grigoriev Vorobjov, Heintz Solerno R., Basu Pollack R.**) algorithms with singly exponential complexity

all based on **Canny**'s geometric idea : construction of an one-dimensional semi-algebraic subset, called a *roadmap*, connected inside every connected component

construction of the roadmap based on recursive calls to itself on several $(O(d)^k)$ in $(k-1)$ dimensional slices, each obtained by fixing the first coordinate at a "critical level"

fixing one coordinate (as a zero of a polynomial of degree $O(d)^k$) produces a polynomial of degree d in $k-1$ variables

$d^{O(k^2)}$ number of recursive calls in the roadmap algorithm :

depth of the recursion k

$$d^{O(k^2)} = \underbrace{d^{O(k)} \times \cdots \times d^{O(k)}}_{k \text{ times}}$$

genericity issues ... several contributions ...

finally, algorithm with complexity $d^{O(k^2)}$ for a general algebraic set (**Basu Pollack R.**), using infinitesimal deformation

deformations, useful for complexity, introduce infinitesimals

now R a real closed field (such as the field of real numbers $\mathbb{R}$) but not necessarily archimedean (such as the field of real Puiseux series $\mathbb{R}\langle\zeta\rangle$)

"connected components" need to be replaced by "semi-algebraically connected components" (for readibility, in this talk we write always "connected component")

Deformation technique explained by an example

Let $Q \in R[X_1, X_2, X_3]$ be defined by

$$Q = X_2^2 - X_1^2 + X_1^4 + X_2^4 + X_3^4.$$

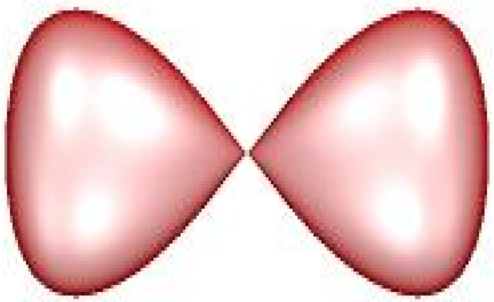

Figure 4.

deformation

$$\mathrm{Def}(Q,\zeta)=Q^2-\zeta\,(X_1^{10}+X_2^{10}+X_3^{10}+1).$$

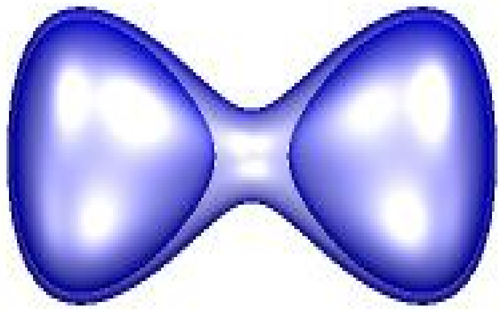

Figure 5.

semi-algebraic set defined by $\mathrm{Def}(Q,\zeta)\leq 0$ (i.e. the part inside the larger component but outside the smaller ones) homotopy equivalent to $\mathrm{Zer}(Q,\mathbb{R}^3)$

3- Motivation to improve classical roadmaps

So, complexity of roadmaps $d^{O(k^2)}$....

Some motivation behind trying to improve this result

- number of connected components of an algebraic set $\mathrm{Zer}(Q, R^k)$ is bounded by $O(d)^k$ where $d = \deg(Q)$
- algorithms for testing emptiness and for computing the Euler-Poincaré characteristic with complexity $d^{O(k)}$
- **D'Acunto, Kurdyka** : geodesic diameter of any connected component of a real variety (defined by polynomials of degree d) contained in an unit ball bounded by $d^{O(k)}$
- many other algorithms in real algebraic geometry use roadmap construction as an intermediate step (i.e. describing connected components, computing higher Betti numbers)

Is there a way to get rid of this gap ?

rather difficult problem with no progress till very recently

4- Baby-step giant-step roadmap

recent method for roadmaps, proposed by **Safey** and **Schost**

applied successfully by them to smooth real algebraic hypersurfaces

new recursive scheme, dimension drops by $\sqrt{k}$ in each recursive call

depth of the recursive calls at most $\sqrt{k}$

complexity of $d^{O(k\sqrt{k})}$

$$d^{O(k\sqrt{k})} = \underbrace{d^{O(k)} \times \cdots \times d^{O(k)}}_{\sqrt{k}\text{ times}}$$

techniques used : complex algebraic geometry, smoothness of polar varieties for generic projections

generic coordinates necessary : non-singularity of polar varieties only true for a Zariski-dense set

important restriction, no known method for making such a choice of generic coordinates deterministically within this improved complexity bound

randomized (rather than deterministic) algorithm for computing roadmaps

possibly :cases where the algorithm terminates and gives a wrong result in contrast, algorithm for constructing classical roadmaps in **Basu Pollack R.** semi-algebraic in nature.

greater flexibility of semi-algebraic geometry :take sums of squares (one single equation for any algebraic set (!), avoids genericity requirements for coordinates

technique used in **Basu, Pollack, R.** : infinitesimal deformation of the given variety so that the original coordinates are good

infinitesimal deformation uses only one infinitesimal

does not affect the asymptotic complexity class

Combine both approaches !

Baby-step giant-step results

main result of **Basu, R., Safey, Schost**

D ordered domain contained in a real closed field R
$V \subset R^k$ zero set of a polynomial of degree $\leqslant d$ with coefficients in D

Theorem 6.

1. *algorithm for constructing a roadmap for V using $d^{O(k\sqrt{k})}$ arithmetic operations in D*
2. *algorithm for counting the number of connected components of V using $d^{O(k\sqrt{k})}$ arithmetic operations in D.*
3. *algorithm for deciding whether two given points belong to the same connected component of V using $d^{O(k\sqrt{k})}$ arithmetic operations in D.*

-points described by real univariate representations of degree $\leqslant d^{O(k)}$) (see **Basu, Pollack, R.**
-general real closed fields natural framework, because use of deformations, necessary for complexity issues

The baby-step giant-step method

main difference between classical roadmap algorithms and baby-step giant-step roadmap algorithms

$\mathrm{Zer}(Q, R^k)$ bounded hypersurface

instead of considering curves in the X_1-direction (think of a finite number of X_2-critical points, above each $x \in R$) and making recursive calls at hyperplane sections of $\mathrm{Zer}(Q, R^k)$ corresponding to critical values of X_1, so that the dimension of the ambient space drops by 1

consider a p-dimensional subset V^0 of $\mathrm{Zer}(Q, R^k)$ where $1 \leq p \leq k$ (think of a finite number of X_{p+1}-critical points, above each $y \in R^p$) and make recursive calls at $(k-p)$-dimensional affine spaces intersected with $\mathrm{Zer}(Q, R^k)$, so that the dimension of the ambient space drops by p

special (generic) situation

$$(V, \mathcal{M}, p, V^0, \mathcal{M}^0)$$

such that

1. $V = \mathrm{Zer}(Q, R^k)$ bounded hypersurface, such that the critical points of the map π_1 on V form the finite set $\mathcal{M} \subset V$;

2. $V^0 \subset V$ closed semi-algebraic set of dimension p, $1 \leq p < k$, such that for each $y \in R^p$, V^0_y finite set of points having non-empty intersection with every connected component of V_y;

3. $\mathcal{M}^0 \subset V$ finite such that the intersection of $\mathcal{M}^0$ with every connected component of V^0_a is non-empty, for $a \in \mathcal{D}^0 = \pi_1(\mathcal{M}^0)$. Moreover for every interval $[a, b]$ and $c \in [a, b]$ with $\{c\} \supset \mathcal{D}^0 \cap [a, b]$, if D is a connected component of $V^0_{[a,b]}$, then D_c is a connected component of V^0_c.

key connectivity result in **Basu, R., Safey, Schost**, generalizing Proposition 4 and a result by **Safey** and **Schost**

Proposition 7. *Let* $\mathcal{N} = \pi_{[1,p]}(\mathcal{M} \cup \mathcal{M}^0)$
$(V, V^0, V_{\mathcal{N}})$ *has good connectivity property*

in order to produce a roadmap of $\mathrm{Zer}(Q, R^k)$

baby-steps: compute a classical roadmap of V^0 passing through $V^0_{\mathcal{N}}$, computed by an algorithm directly adapted from **Basu, Pollack, R.**, using that V^0 is special (zero-dimensional fibers parametrized by R^p, so that the depth of the recursion is p), in time $d^{O(pk)}$

giant-steps compute the roadmap of a finite set of fibers in a $(k-p)$-dimensional ambient space, using recursive calls

at each recursive call, hypersurface, in a smaller dimensional ambiant space general case uses sums of squares, a deformation technique and a limit process

fixing $p = \sqrt{k}$ is the optimal choice

Main ideas clear and straightforward

many technical difficulties to reach the complexity $d^{O(k\sqrt{k})}$ for a general algebraic set

fixing a whole block of $\sqrt{k}$ variables at a time necessitates introducing a new kind of algebraic representation, called "real block representation"... complexity issues

limit process ... for a point no problem, for a curve more complicated .... complexity issues

Basu,R., Safey, Schost very long 50 pages paper

4- Divide and conquer

what next ? divide and conquer very natural idea

$S = \mathrm{Zer}(Q, R^k)$ general algebraic set

consider a $k/2$-dimensional subset S^0 of S (think of a finite number of critical points, above each $y \in R^{k/2}$) and make recursive calls
- at S^0 itself
- at S^1, union of certain $(k/2)$-dimensional linear spaces intersected with S

prove that (S, S^0, S^1) have good connectivity property

main difficulty to overcome: in recursive calls, no more an hypersurface in a smaller ambiant space but algebraic sets of various codimensions (even if the starting point is an hypersurface)

even if the original situation is sufficiently generic, such genericity properties are difficult to maintain throughout the algorithm ...

Two approaches

-work in progress of **Safey** and **Schost** in the case of a smooth hypersurface (probabilistic) using polar varieties

-work in progress of **Basu** and **R.** in the case of a general real algebraic set, using deformations and semi-algebraic techniques

Some indications on Basu R. approach

Definition of a G-roadmap

for genericity reasons, introduce G polynomial function from R^k to R (generalizing projection on X_1)

Definition 8. *A G-roadmap for S is a semi-algebraic set M of dimension at most one contained in S which satisfies the following roadmap conditions:*

- RM_1 *For every semi-algebraically connected component D of S, $D \cap M$ is semi-algebraically connected.*
- RM_2 *For every $x \in R$ and for every semi-algebraically connected component D' of $S \cap G^{-1}(\{x\})$, $D' \cap M \neq \emptyset$.*

Statement

Theorem 9.
Let V be the zero set of a polynomials of degree d in k variables and with coefficients in an ordered domain D. We describe

1. *algorithm for constructing a G-roadmap for V using $\left(k^{\log(k)}\,d\right)^{k\log^2(k)}$ arithmetic operations in D*
2. *an algorithm for counting the number of connected components of V using $\left(k^{\log(k)}\,d\right)^{k\log^2(k)}$ arithmetic operations in D.*
3. *an algorithm for deciding whether two given points belong to the same connected component of V using $\left(k^{\log(k)}\,d\right)^{k\log^2(k)}$ arithmetic operations in D.*

Connectivity result

$1 \leq q < p \leq k$, $G \in R[X_1, \ldots, X_k]$, and $\mathcal{P}, \mathcal{Q} \subset R[X_1, \ldots, X_k]$ be finite, $\mathrm{card}(\mathcal{P}) = k - p$,

$$S = \{x \in R^k \mid P(x) = 0, Q(x) \geq 0, \forall P \in \mathcal{P}, \forall Q \in \mathcal{Q}\}$$

special situation

$$(S, \mathcal{M}, p, q, \mathcal{S}^0, \mathcal{M}^0)$$

such that

1. $\mathcal{M}$ finite set of critical points of G on S (union of critical points of G for every zero set of $\mathcal{P}' \subset \mathcal{P} \cup \mathcal{Q}$).

2. $S^0 \subset S$ q-dimensional semi-algebraic set such that for each for each $y \in R^{[1,q]}$, $S^0_y := S^0 \cap \pi^{-1}_{[1,q]}(y)$ finite, meets every connected component of $S_y := S \cap \pi^{-1}_{[1,q]}(y)$, and contains a minimizer of G over each connected component of S_y.

3. $\mathcal{M}^0 \subset S^0$ finite, meets every connected component of $S^0_{G=a}$ for each $a \in \mathcal{D}^0 := G(\mathcal{M}^0)$. Moreover for every interval $[a, b] \subset R$ and $c \in [a, b]$, with $\{c\} \supset \mathcal{D}^0 \cap [a, b]$, if D is a connected component of $S^0_{a \leq G \leq b}$, then $D_{G=c}$ is a connected component of $S^0_{G=c}$.

key connectivity result, generalizing Proposition 7 (with a quite similar proof)

Proposition 10. *Let* $\mathcal{N} := \pi_{[1,q]}(\mathcal{M} \cup \mathcal{M}^0)$

$(S, S^0, S_{\mathcal{N}})$ *has good connectivity property with respect to* S.

In divide and conquer we take $p = k/2$,
rather than $p = \sqrt{k}$ as in baby-giant.

Divide and conquer roadmap consruction

in order to produce a G-roadmap of $\mathrm{Zer}(Q, R^k)$

ideally
divide and conquer: make recursive calls to S^0 (critical points) and $S^1 = S_{\mathcal{N}}$(well chosen fibers of preceeding set), each of half dimension
take critical points of critical points etc...

but
need to make deformations to ensure genericity properties
no more sums of squares (general co dimensions)
use slighly different deformation technique coming from **Jeronimo, Perrucci** and **Tsigaridas** to reach **special situation**
add four infinitesimals at each level of the recursion to ensure genericity
take critical points of deformation of critical points
introduce charts for having good equations of the critical locus

Complexity analysis (sketch)

depth of recursion : $\log(k)$

number of recursive calls : $\left(k^{\log(k)}\, d\right)^k$

number of infinitesimals introduced :$O(\log(k))$

maximal degree with respect to X : $k^{\log(k)}\, d$

number of arithmetic operations

$$\left(k^{\log(k)}\, d\right)^{k \log^2(k)}$$

complexity issues non trivial ...

hope for the length of a semi-algebraic path

$$\left(k^{\log(k)}\, d\right)^{k \log(k)}$$

(cf. **D'Acunto** and **Kurdyka**)

Open problems

supposing that the divide and conquer roadmap for general algebraic sets works

doable

- ”short” semi-algebraic path
- divide and conquet semi-algebraic sets rather than algebraic sets

to explore, improvements on

- description of connected components (parametrized roadmap)
- computation of higher Betti numbers

no ideas so far but ...

- special case of quadratic sets (improved bounds on number of connected components, geodesic diameter), what about roadmap ?

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