Bad words arising from generalized Fibonacci cubes

Sandi Klavžar
University of Ljubljana, Slovenia
University of Maribor, Slovenia
joint work with *Aleksandar Ilić, Yoomi Rho, Sergey Shpectorov*

Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence
CNRS, LaBRI, UMR5800, F-33400 Talence
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Fibonacci cubes

Hypercubes

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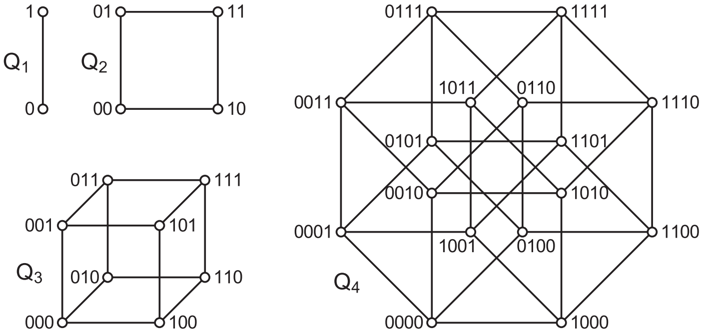
- Q_d is d-regular
- Q_d is vertex-transitive ...

Small cubes


Q1
1
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Q2
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11
00
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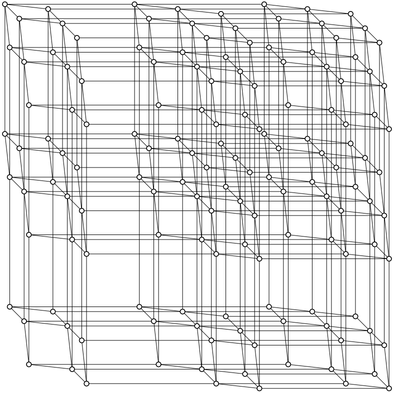
Introduced in (Hsu, 1993) as a model for interconnection networks.

The Fibonacci cube Γ_{10}



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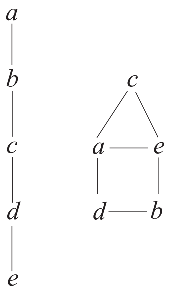

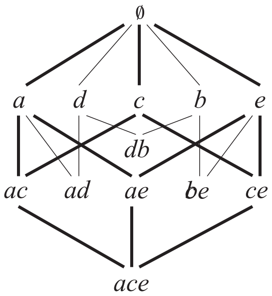

Selected results on F_d

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The number of 4-cycles of Γ_n (K., 2005):

$$-\frac{3n}{25}F_{n+1} + \left(\frac{n^2}{10} + \frac{3n}{50} - \frac{1}{25}\right)F_n\,.$$

Selected results on F_d cont'd

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(K., Mollard, Petkovšek, 2011) The number of vertices of Γ_n *having degree* k *is* $\sum_{i=0}^{k} \binom{n-2i}{k-i}\binom{i+1}{n-k-i+1}$.

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(K., Mollard, 2012) $W(\Gamma_n) = \frac{4(n+1)F_n^2}{25} + \frac{(9n+2)F_nF_{n+1}}{25} + \frac{6nF_{n+1}^2}{25}$.

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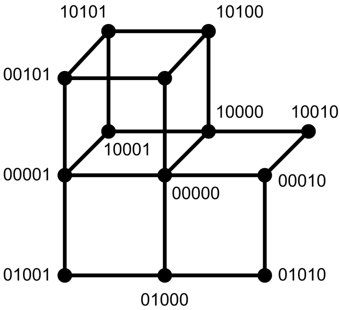
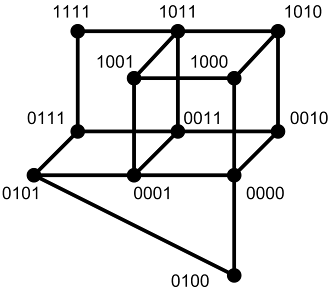
- The generalized Fibonacci cube, $Q_d(f)$, is the graph obtained from Q_d by removing all vertices that contain f as a substring.
- Note: $\Gamma_d = Q_d(11)$.

Fibonacci cube $Q_5(11)$ and 110-Fibonacci cube $Q_4(110)$





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$$|S(H_d)| = -\frac{3(d+1)}{25} F_{d+2} + \left(\frac{(d+1)^2}{10} + \frac{3(d+1)}{50} - \frac{1}{25} \right) F_{d+1}.$$

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Problem

For which f and d, $Q_d(f) \hookrightarrow Q_d$ holds?

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Theorem

Let $d \geq 2$*. Then*

(i) *For* $r \geq 1$, $Q_d(1^r 0) \hookrightarrow Q_d$.

(ii) *For* $s \geq 2$, $Q_d(1^2 0^s) \hookrightarrow Q_d$ *if and only if* $d \leq s + 4$.

(iii) *If* $r, s \geq 3$, *then* $Q_d(1^r 0^s) \hookrightarrow Q_d$ *if and only if* $d \leq 2r + 2s - 3$.

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1: Good and bad words

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Lemma

Suppose that $Q_d(f) \not\hookrightarrow Q_d$ for some dimension d. Then $Q_{d'}(f) \not\hookrightarrow Q_{d'}$ for all $d' \geq d$.

Bad words cont'd

Proof

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- $\mathcal{T}_n^s$ ... split words from $\mathcal{T}_n$.

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Theorem

$$\lim_{n\to\infty} \frac{|\mathcal{B}_n|}{2^n} = a\,.$$

Estimates for the limit value a

n	$\lvert\mathcal{T}_n^s\rvert$	$\lvert\mathcal{T}_n^s\rvert/2^n$
4	4	0.250000
6	34	0.531250
8	182	0.710938
10	830	0.810547
12	3518	0.858887
14	14538	0.887329
16	59074	0.901398
18	238534	0.909935
22	3845886	0.916931
24	15408114	0.918395
26	61689006	0.919238
28	246881258	0.919704
30	987815218	0.919975

n	$\|\mathcal{T}_n\|$	$\|\mathcal{T}_n\|/2^n$
4	8	0.500000
5	22	0.687500
6	46	0.718750
7	98	0.765625
8	210	0.820313
9	430	0.839844
10	886	0.865234
25	30873042	0.920088
26	61759618	0.920290
27	123512490	0.920240
28	247051278	0.920338
29	494077866	0.920292
30	988213906	0.920346
31	1976359510	0.920314

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Proof cont'd

- Let $\mu_n = T_n/2^n$ (notice $\mu_n = 1 - a_n$), then

$$\mu_n \leq \mu_{n+2} + \frac{k(k+1)}{2^{k+2}}\,.$$

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$$\mu_{n+2} \geq \mu_n - \frac{k(k+1)}{2^{k+2}}.$$

- Since $a_n = 1 - \mu_n$ we get

$$a_{n+2} \leq a_n + \frac{k(k+1)}{2^{k+2}}.$$

Proof cont'd

- Combining these relations from n to $n+2m$ we get

$$a_{n+2m} \leq a_n + \sum_{i=k}^{k+m-1} \frac{i(i+1)}{2^{i+2}} \, .$$

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- Using computer with $k = 15$ we get that the sum here is at most 0.004181.
- Together with the value $a_{30} = 0.919975$ this yields an upper limit of 0.924156.

2: Index of binary words

Index

Definition

The index of a word f, denoted $\beta(f)$, is the smallest integer d such that $Q_d(f) \not\hookrightarrow Q_d$. If no such integer exists we set $\beta(f) = \infty$.

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Clearly, $\beta(f) < \infty$ if and only if f is bad.

Theorem

Let f be a bad word. Then $\beta(f) < |f|^2$.

Index cont'd

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For almost all bad words f, $\beta(f) < 2|f|$.

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Conjecture verified by computer for all words of length at most 10 and dimension at most 20.

3: Parity index of binary words

Motivation - hamiltonicity

Recall: Γ_d has a hamiltonian path for any $d \geq 0$.

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(Liu, Hsu, Chung, 1994) Each $Q_d(1^r)$ contains a hamiltonian path.

Question: what about $Q_d(f)$?

Parity index defined

- Even/odd words of $Q_d(f)$ form its bipartition.

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- $\Delta(X) = e(X) - o(X)$.
- $\mathrm{PI}_d(f) = \Delta(\mathcal{F}_d(f))$ ... parity index of f of dimension d.

Therefore, a necessary condition for $Q_d(f)$ to contain a hamiltonian path is:

$$|\mathrm{PI}_d(f)| \leq 1\,.$$

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A word b is a power of a word c if $b = c^k$ for some $k \geq 1$.

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Definition

A word b is a power of a word c if $b = c^k$ for some $k \geq 1$.

Theorem

Let f be a power of a prime word. Then $|\mathrm{PI}_d(f)| \leq 1$ for any $d \geq 1$.

Prime words cont'd

Conjecture

Let f be a word such that $|\mathrm{PI}_d(f)| \leq 1$ holds for any d. Then f is a power of a prime word.

Prime words cont'd

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Let f be a word such that $|\mathrm{PI}_d(f)| \le 1$ holds for any d. Then f is a power of a prime word.

Theorem

Let $r \ge 1$. Then

$$|\mathrm{PI}_d(0^r 1 0^r)| = \begin{cases} 0; & d \le 2r, 2r+2 \le d \le 3r+1\,, \\ 1; & d = 2r+1, 3r+2 \le d \le 4r+3\,. \end{cases}$$

Moreover, for any $d \ge 4r+4$, $|\mathrm{PI}_d(0^r 1 0^r)| \ge 2$.

Support for the conjecture

Theorem

Let $r, s, t \geq 1$. Let z be the integer such that $(z-1)t + 2 \leq r + s \leq zt + 1$. Then

$$|\mathrm{PI}_d(0^r 1^s 0^t)| \begin{cases} = 0; & d < r+s+t, \\ & y(r+s+t) < d < (y+1)(r+s)+t; 1 \leq y \leq z\,, \\ \geq 1; & d = r+s+t, \\ & (y+1)(r+s)+t \leq d \leq (y+1)(r+s+t); 1 \leq y \leq z, \\ & d = (z+1)(r+s+t)+1\,. \end{cases}$$

Moreover, for any $d \geq (z+1)(r+s+t)+2$, $|\mathrm{PI}_d(0^r 1^s 0^t)| \geq 2$.

Support for the conjecture cont'd

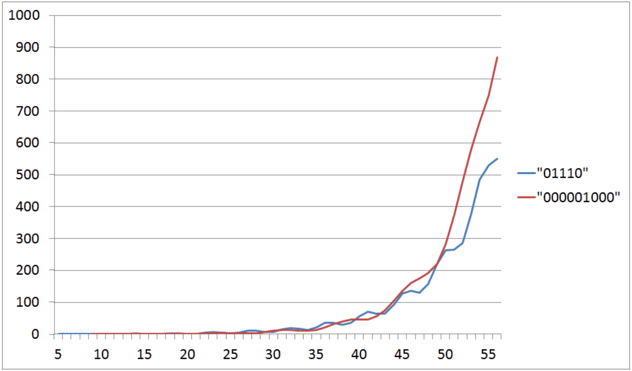
$\mathrm{PI}_d(f)$ computed for $|f| \leq 10$, $d \leq 31$. Balanced words:

Support for the conjecture cont'd

$\mathrm{PI}_d(f)$ computed for $|f| \leq 10$, $d \leq 31$. Balanced words:

length	f
3	001
4	0001, 0011, 0101
5	00001, 00011, 00101
6	000001, 000011, 000101, 000111 001001, 001011, 001101, 010101
7	0000001, 0000011, 0000101, 0000111 0001001, 0001011, 0001101, 0010011 0010101, 0011101
8	00000001, 00000011, 00000101, 00000111 00001001, 00001011, 00001101, 00001111 00010001, 00010011, 00010101, 00010111 00011001, 00011011, 00011101, 00100011 00100101, 00101011, 00101101, 00110011 00110101, 00111101, 01010101

Values of $|\mathrm{PI}_d(f)|$ for $f = 01110$ and $f = 000001000$



Merci beaucoup!