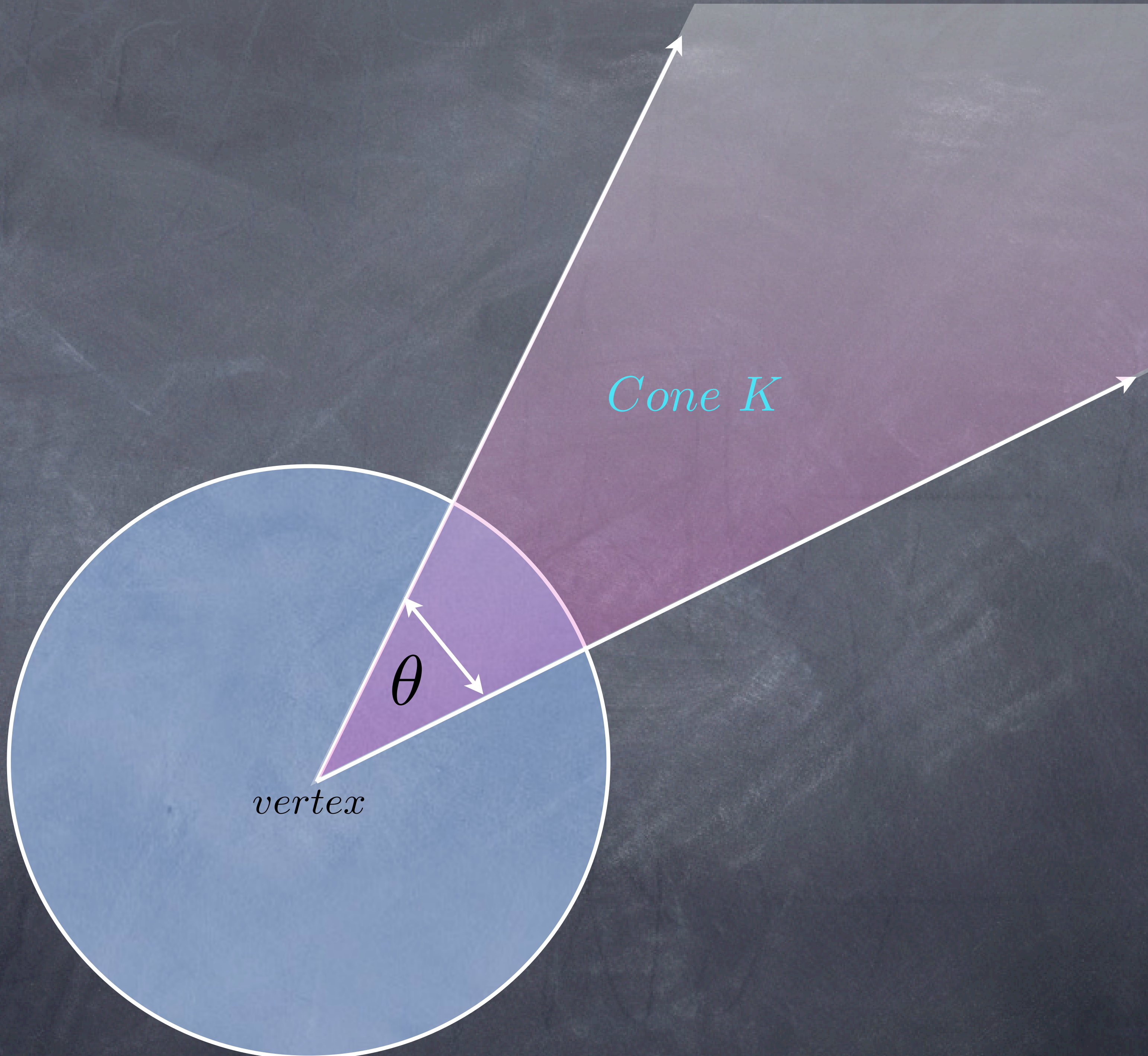
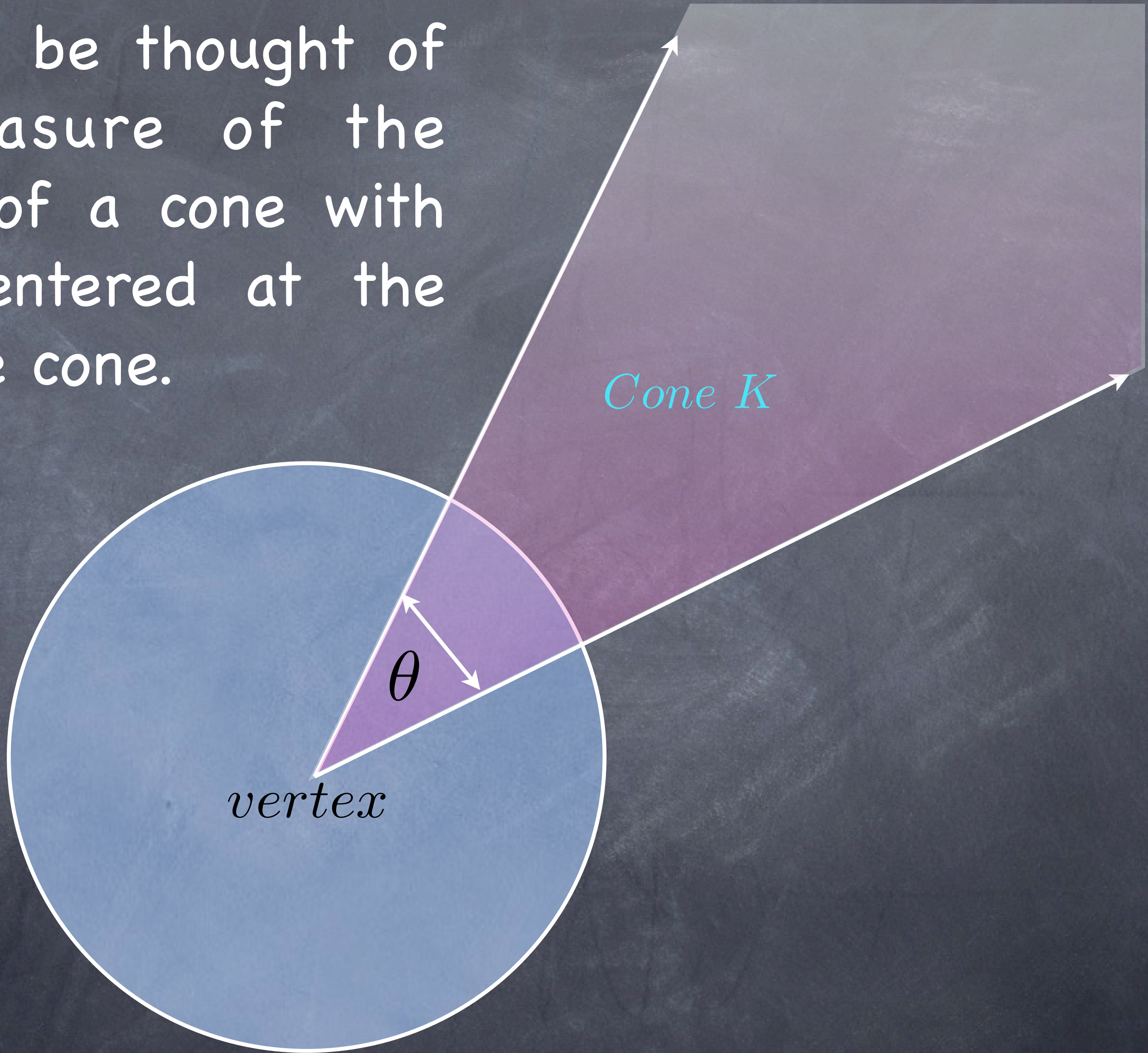
Cone theta functions and what they tell us about the irrationality of spherical polytope volumes

Universite de Bordeaux
Algorithmic Number Theory Seminar
June 2013

Sinai Robins
CNRS/LAAS, Toulouse France
and NTU, Singapore
Joint work with Amanda Folsom and Winfried Kohnen


$Cone\ K$
θ
$vertex$

An angle can be thought of as the measure of the intersection of a cone with a sphere, centered at the vertex of the cone.



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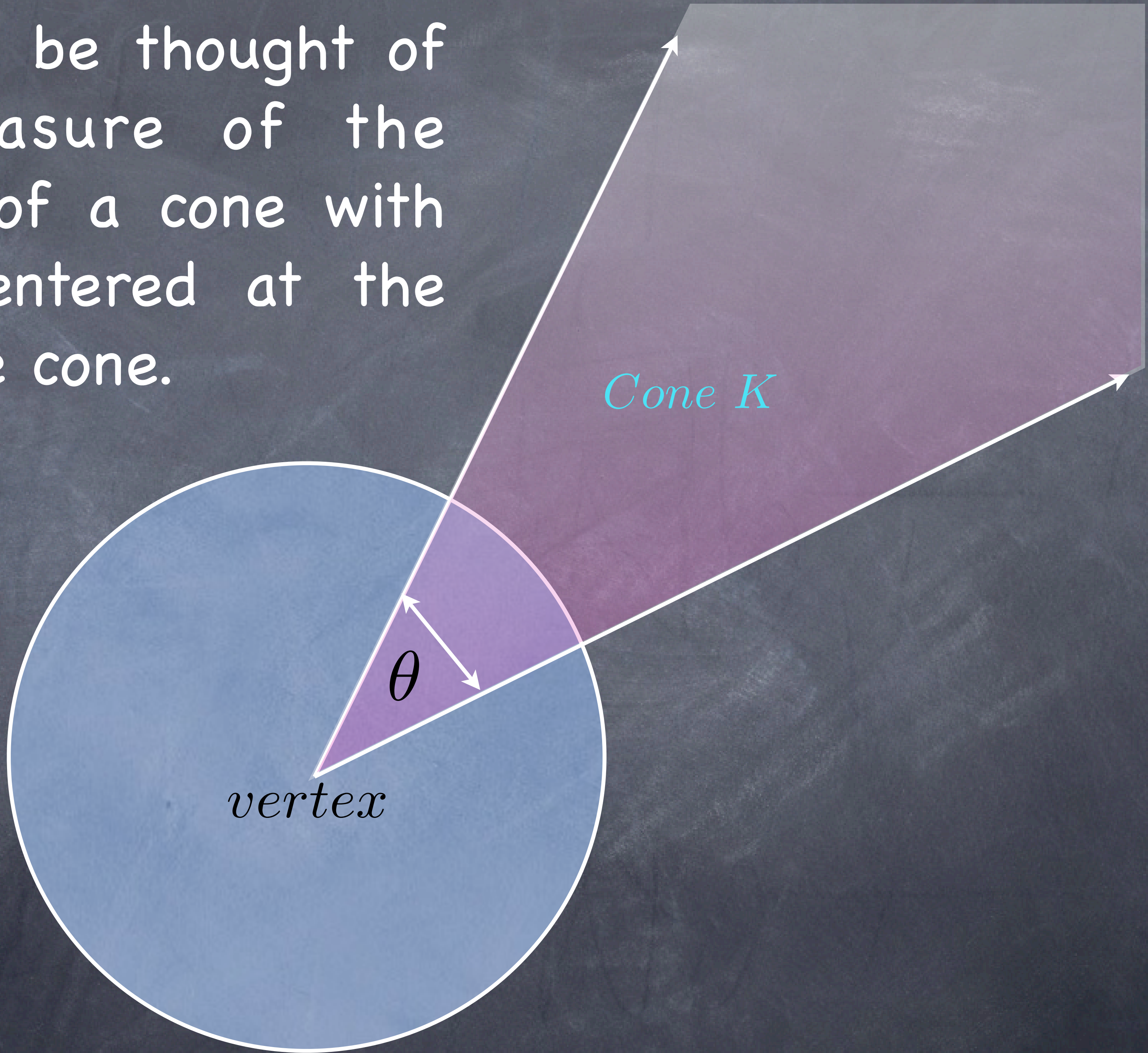
What is a higher-dim'l angle?



A cone $K \subset \mathbb{R}^d$ is the non-negative real span of a finite number number of vectors in Euclidean space.

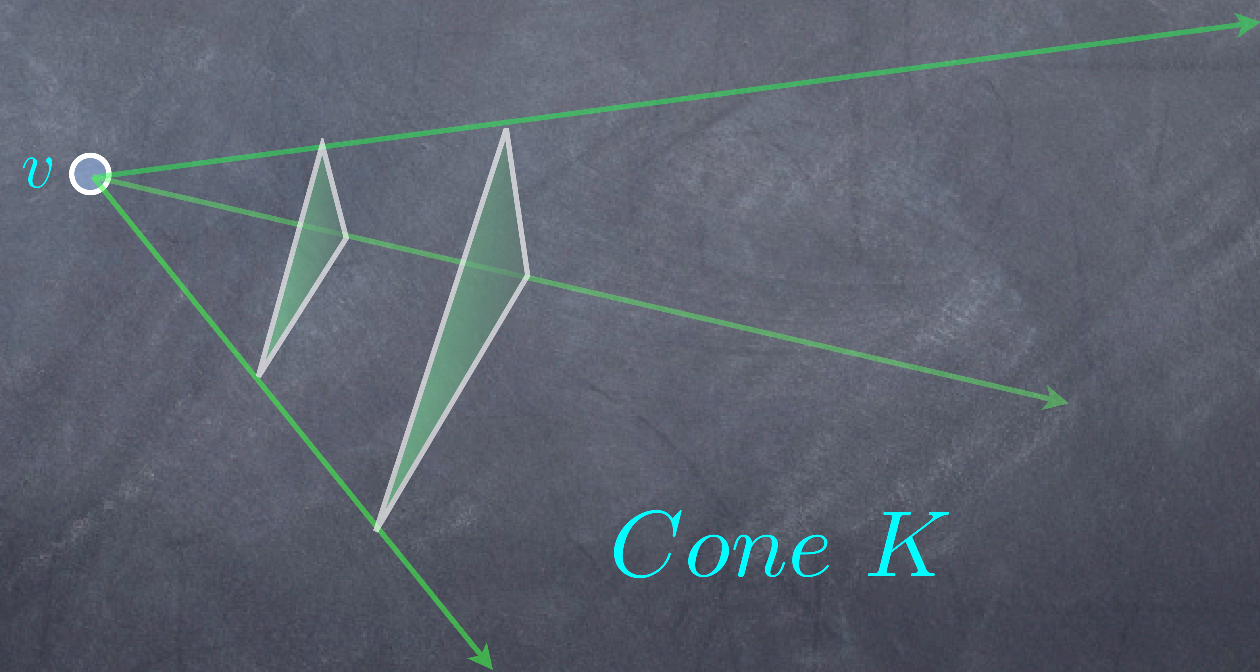
That is, a cone is defined by:

$$K = \{\lambda_1 W_1 + \ldots + \lambda_d W_d \mid all\ \lambda_j \geq 0\}$$

where we assume that the vectors

$W_1, \ldots, W_d$ are linearly independent in $\mathbb{R}^d$.

Example: a 3-dimensional cone.



How do we describe an angle analytically in higher dimensions?

A nice analytic description of an angle is given by:

$$angle \;= \int_K e^{-\pi(x^2+y^2)} dxdy$$

The solid angle at the vertex v of a cone K in $\mathbb{R}^d$ $= \omega_K(v) = \int_K e^{-\pi||x||^2} dx$

A solid angle in dimension d is equivalently:

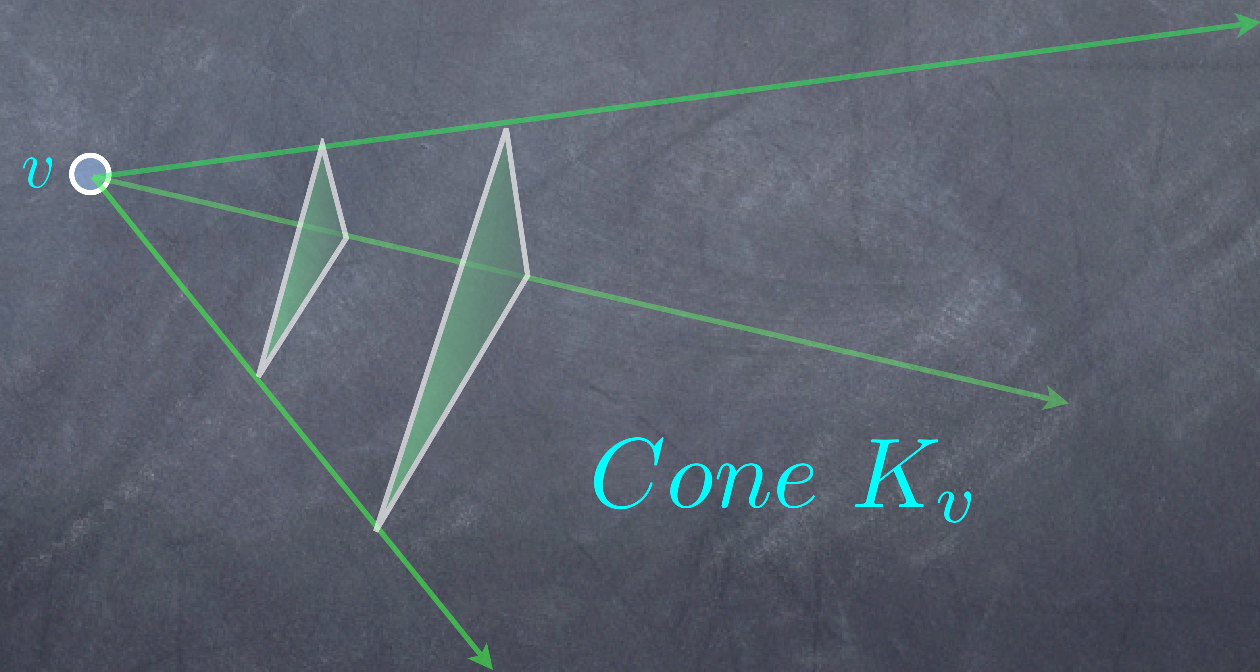
1. The proportion of a sphere, centered at the vertex of a cone, which intersects the cone.

2. The probability that a randomly chosen point in Euclidean space, chosen from a fixed sphere centered at the vertex of K, will lie inside K.

3. $$Solid\ angle = \int_K e^{-\pi(x^2+y^2)} dx dy$$

4. The volume of a spherical polytope.

Example: defining the solid angle at a vertex of a 3-dimensional cone.



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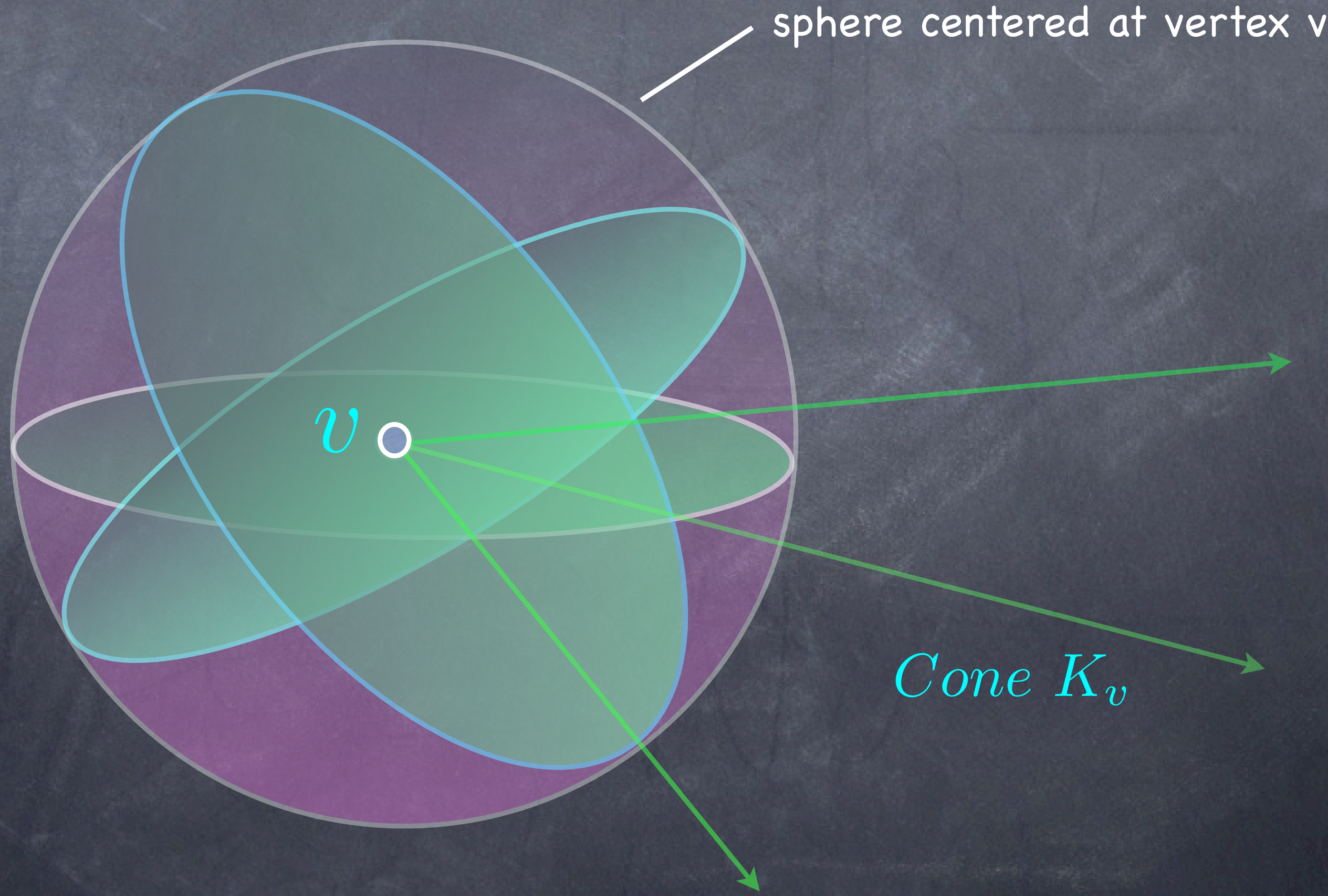

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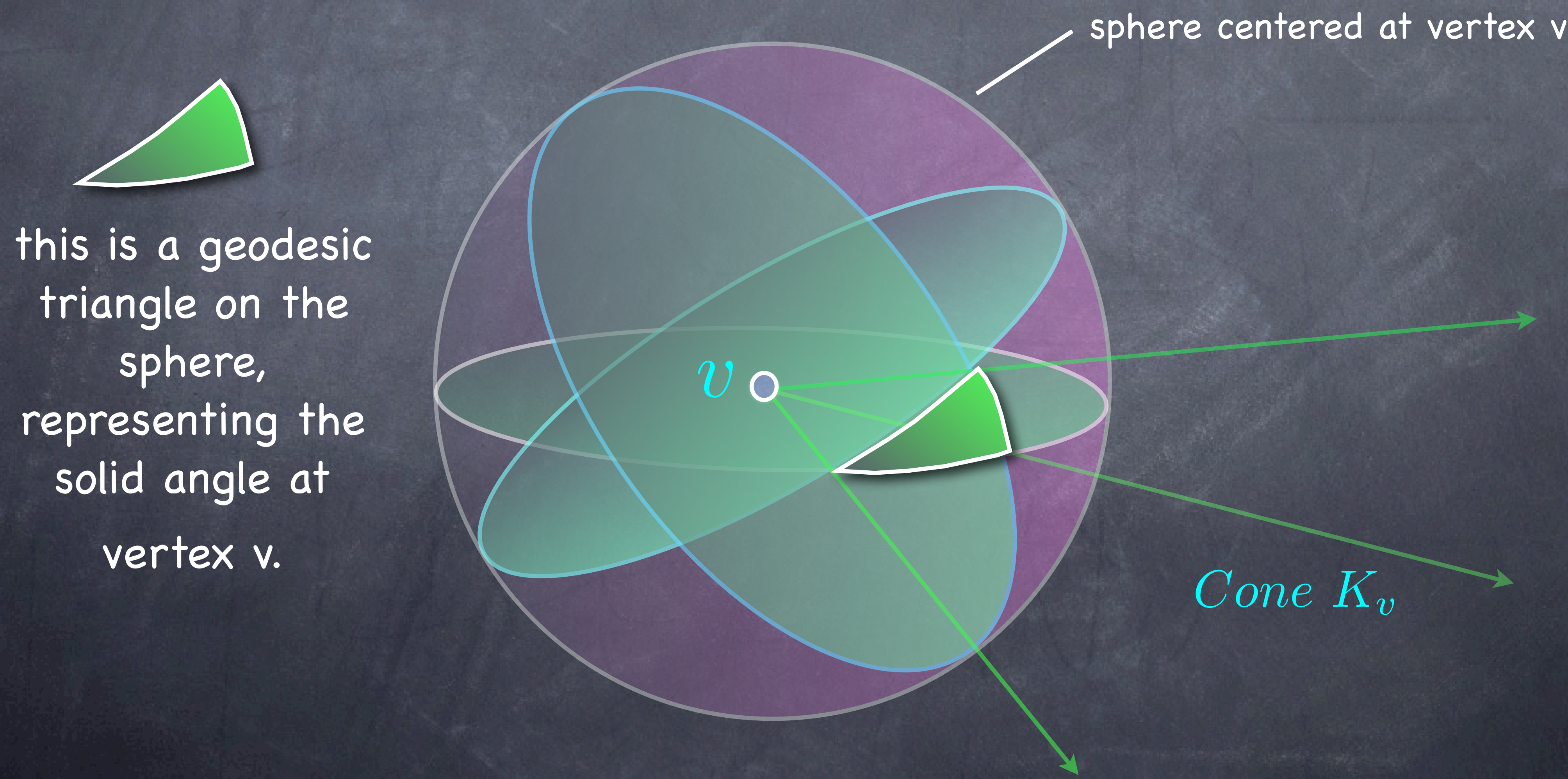


The moral: a solid angle in higher dimensions is really the volume of a spherical polytope.

To help us analyze solid angles, we introduced the following cone theta function for a cone K, and a full rank lattice L:

Definition.

$$\Phi_{K,\mathcal{L}}(\tau) := \sum_{m \in \mathcal{L} \cap \mathcal{K}} e^{\pi i \tau ||m||^2},$$

where τ is in the upper complex half plane.

Example.

For the cone theta function of the positive orthant $K_0 := \mathbb{R}^d_{\geq 0}$, and with $\mathcal{L}$ the integer lattice, we claim that

$$\Phi_{K_0}(\tau) = \frac{1}{2^d}\left(\theta(\tau) + 1\right)^d,$$

where $\theta(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i \tau n^2}$, the classical weight 1/2 modular form. In particular,

$$\Phi_{K_0}(\tau) = \frac{1}{2^d}\sum_{k=0}^{d}\binom{d}{k}\theta^k(\tau),$$

There is an analytic link between solid angles and these conic theta functions, given by:

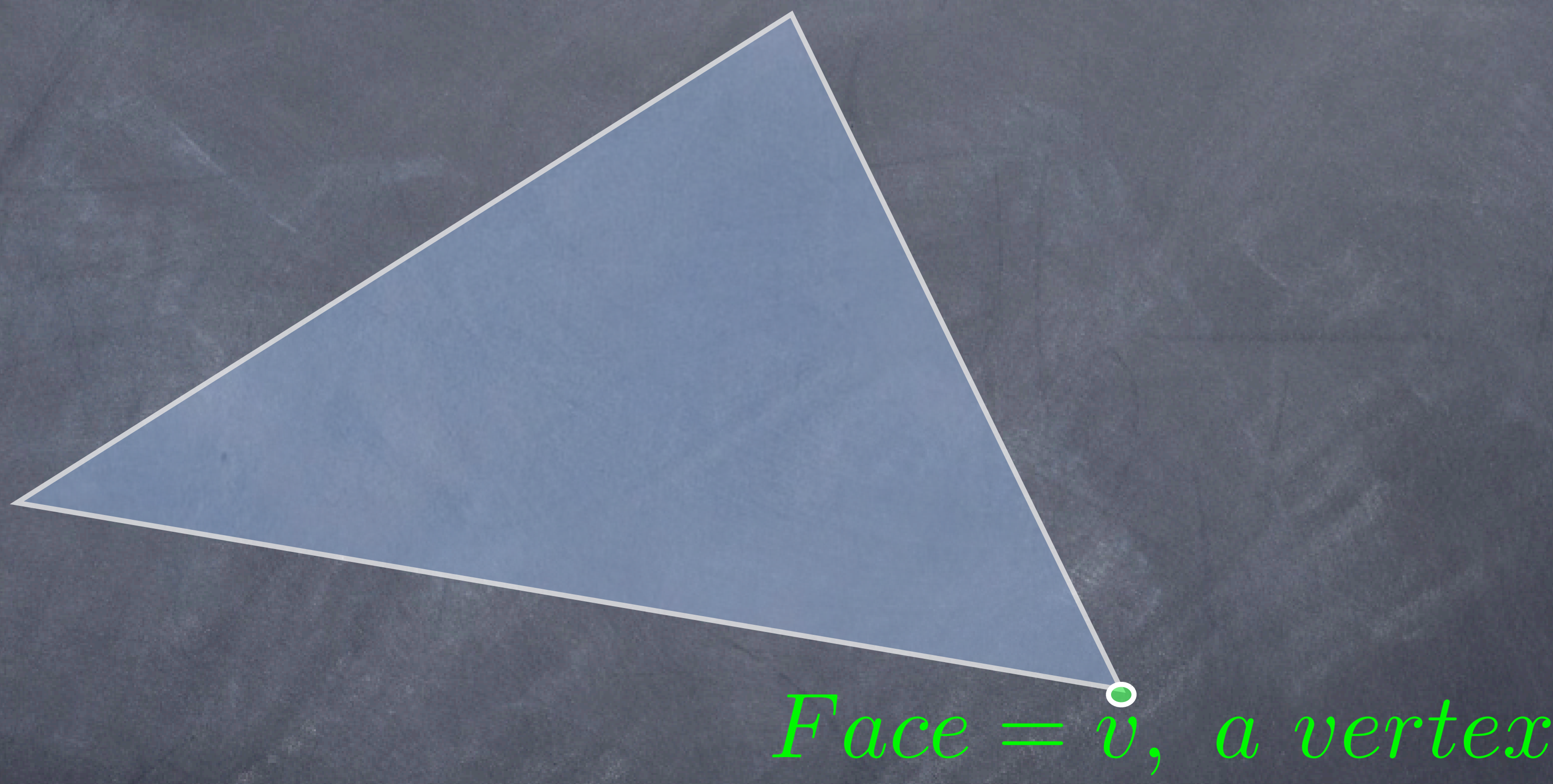
Lemma.

$$t^{\frac{d}{2}} \Phi_{K,\mathcal{L}}(it) \sim \frac{\omega_K}{|detK|},$$

as $t \to 0^+$.

What are tangent cones?

Example: If the face F is a vertex, what does the tangent cone at the vertex look like?



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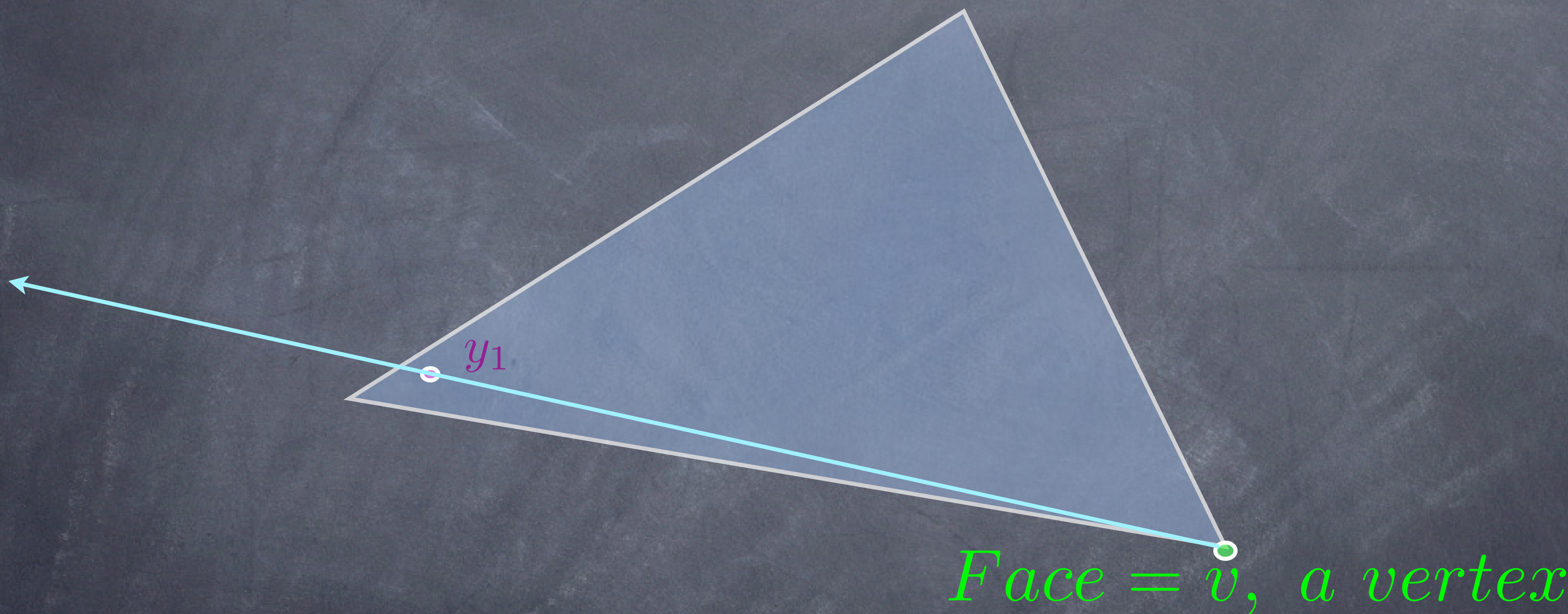
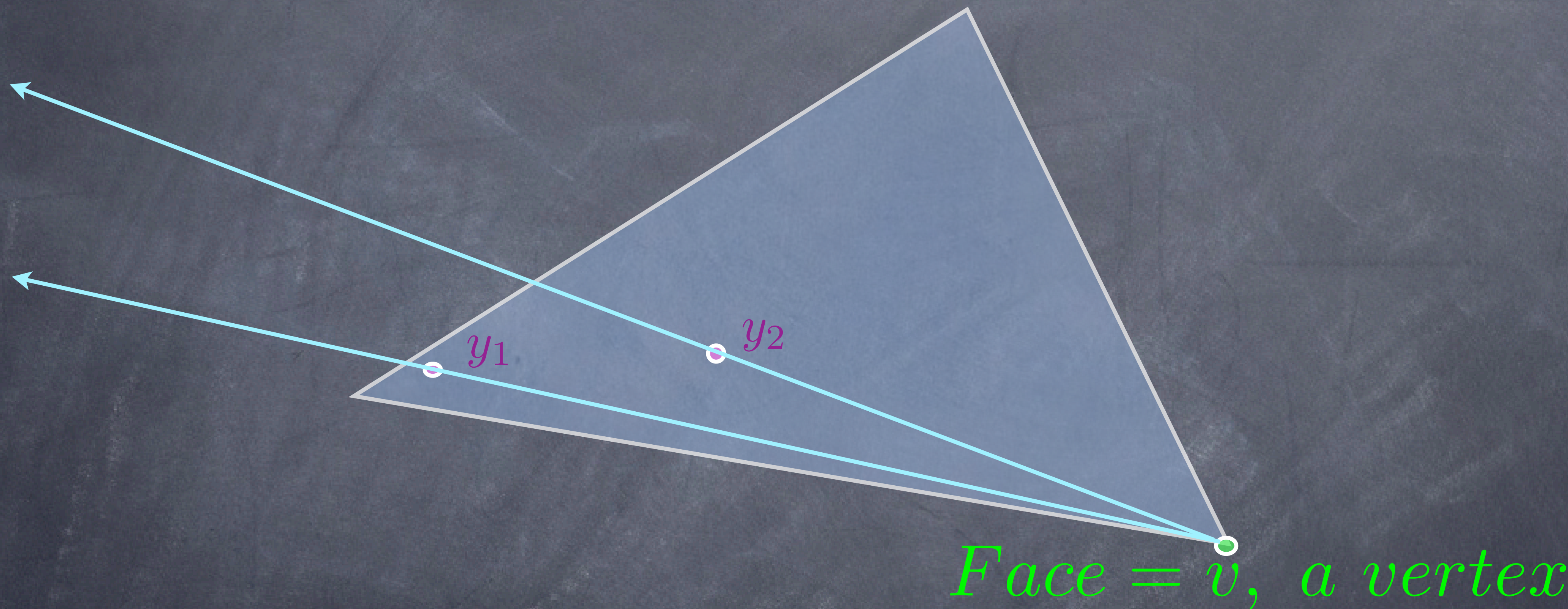
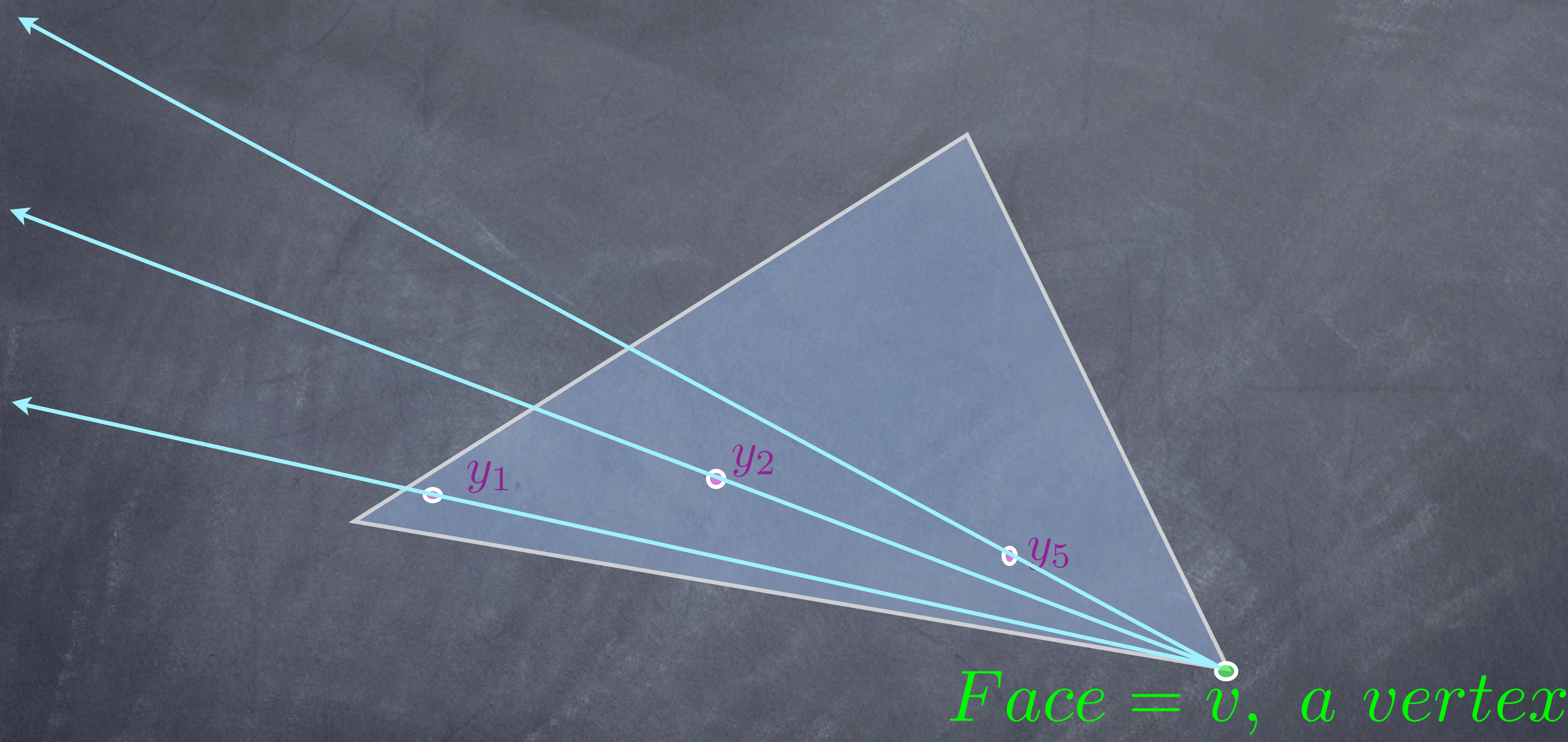
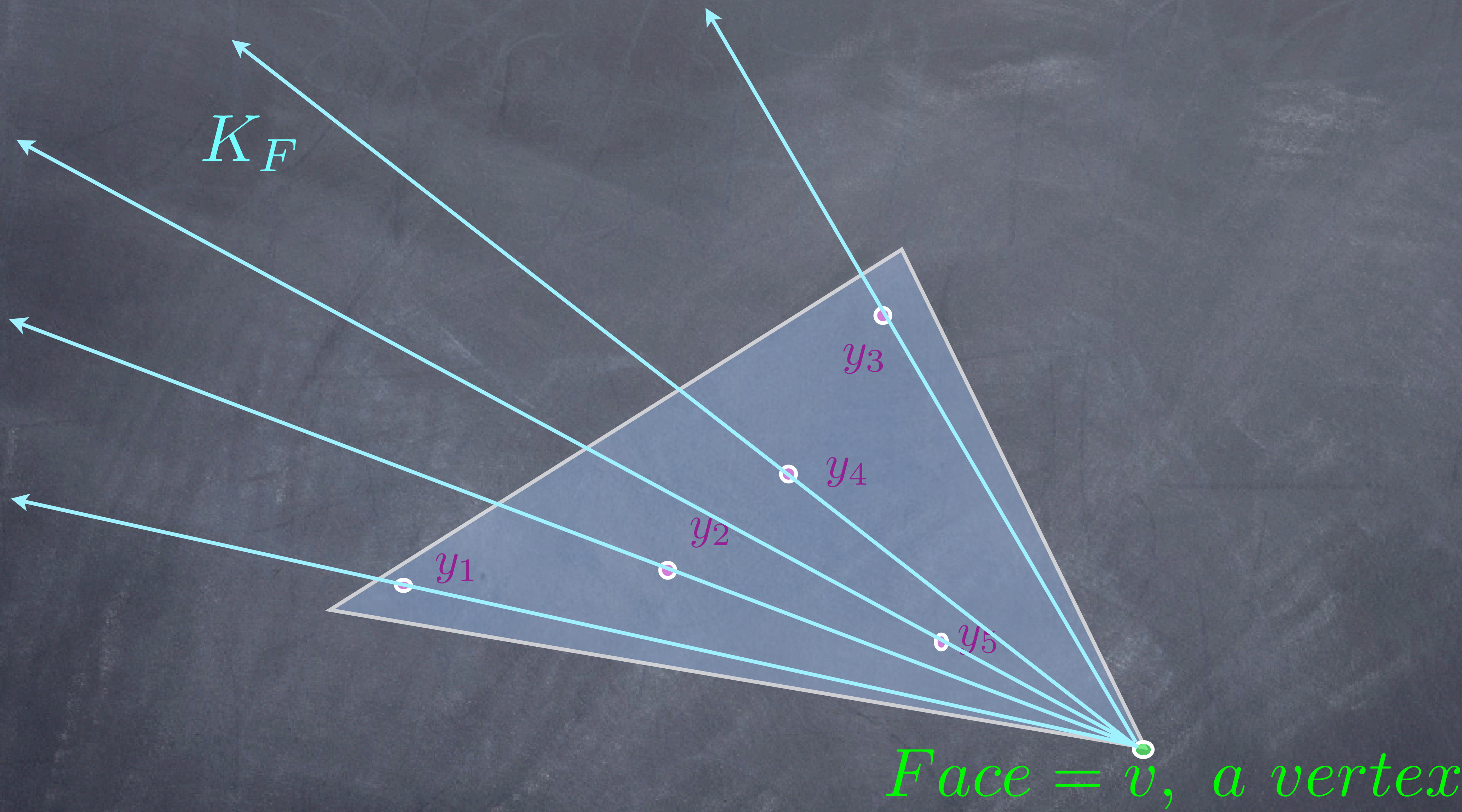
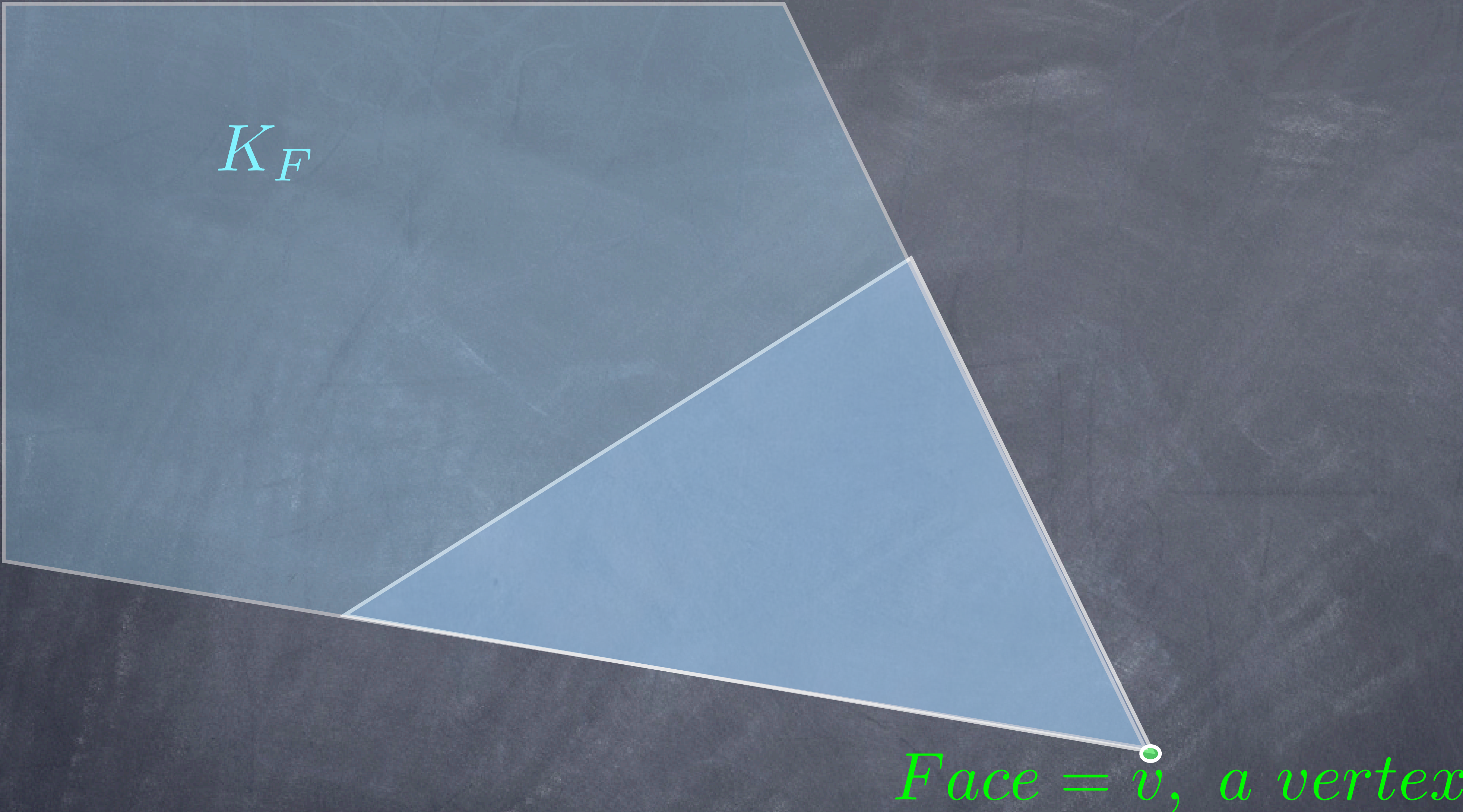



y_1
y_2
$Face = v,\ a\ vertex$


y_1
y_2
y_5
$Face = v,\ a\ vertex$



The tangent cone K_F is the union of all of these rays from the face F = vertex v



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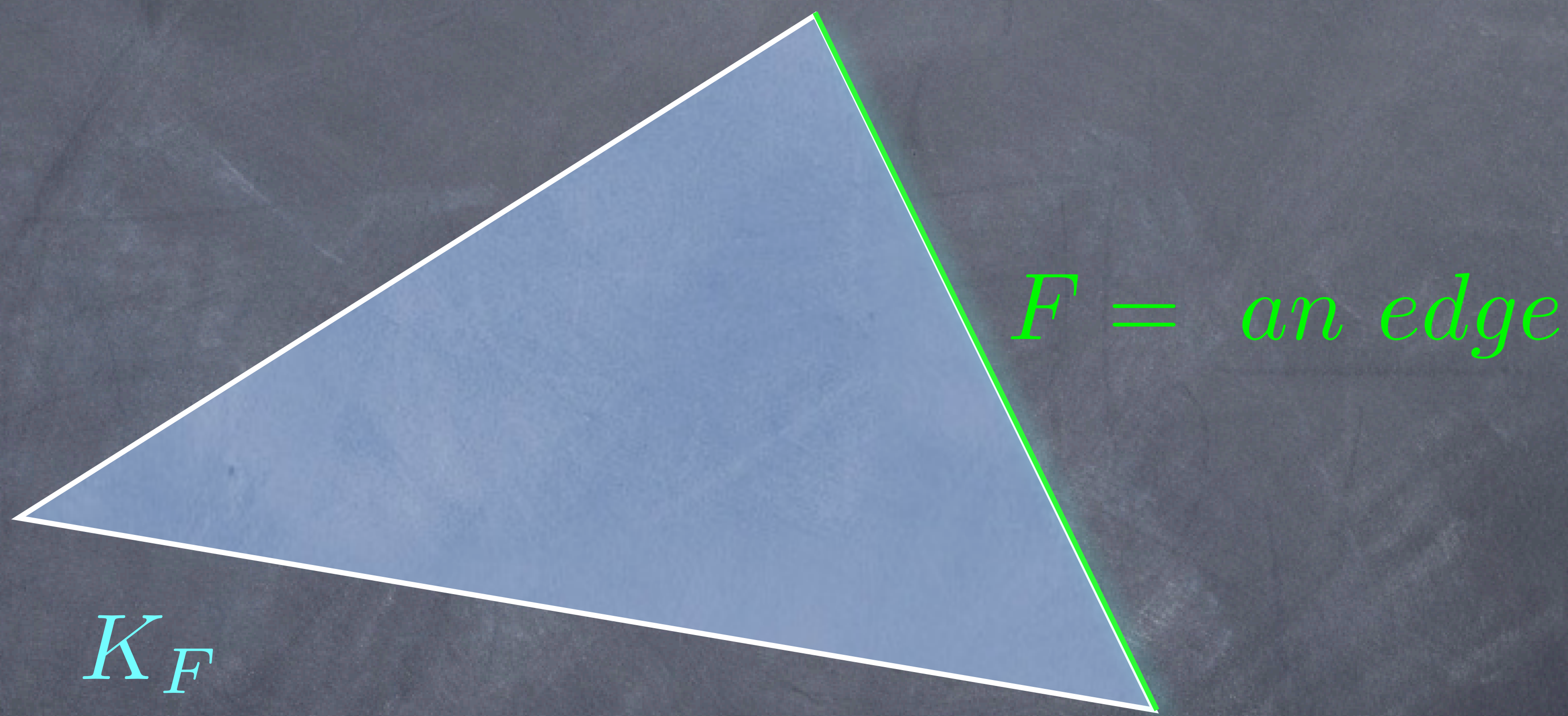
Definition. The tangent cone K_F of a face $F \subset P$ is defined by

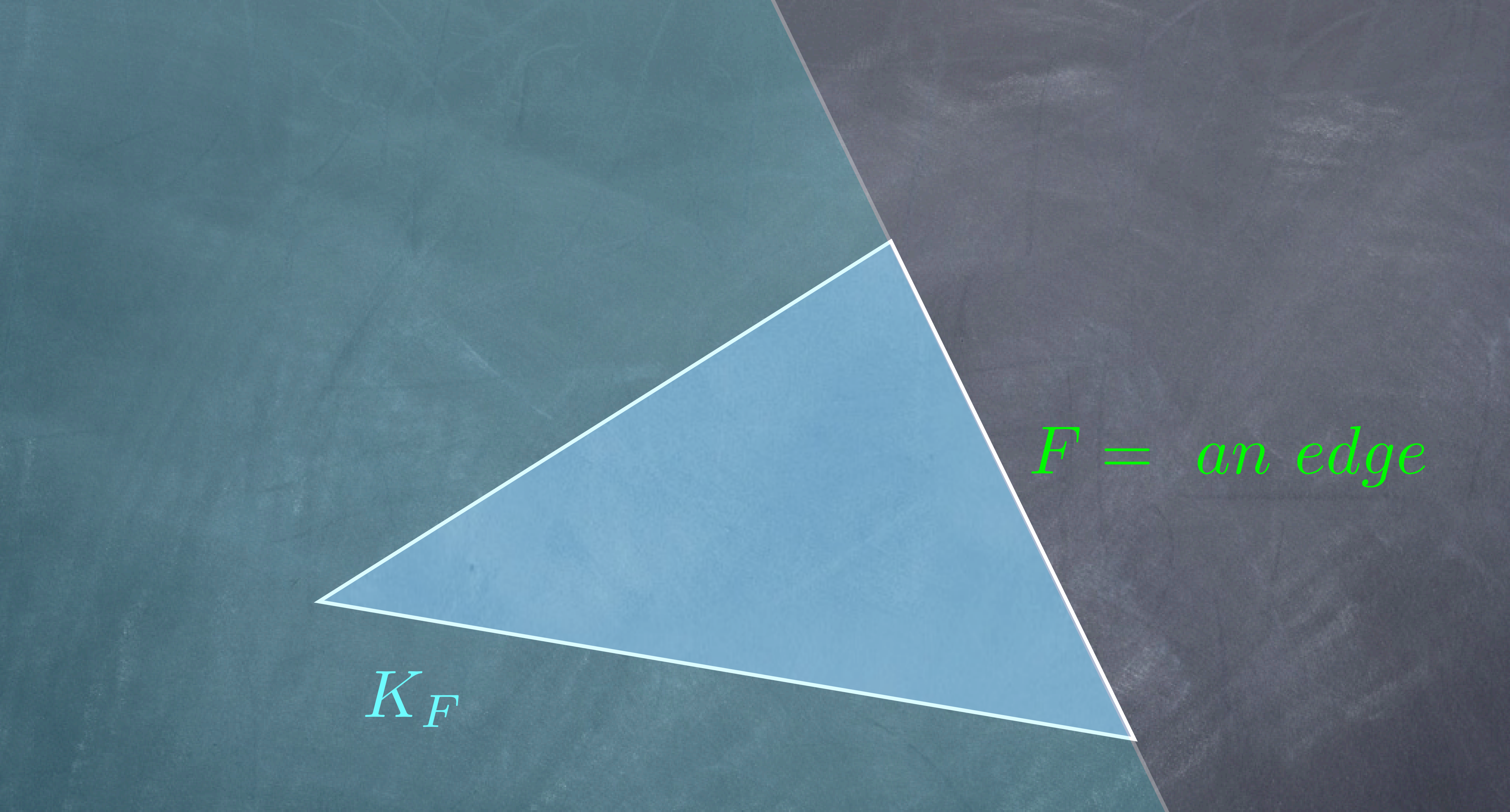
$$K_F = \{x + \lambda(y - x) \mid x \in F, y \in P, \text{ and } \lambda \geq 0\}.$$

Intuitively, the tangent cone of F is the union of all rays that have a base point in F and point 'towards P'.

We note that the tangent cone of F contains the affine span of F.



Example. when the face F is a 1-dimensional edge of a polygon, the tangent cone of F is a half-plane.



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Question 1. Which lattice polyhedral cones K give rise to spherical polytopes with a rational volume?

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Question 2. Analyzing the cone theta function Φ_K attached to a polyhedral cone K, how 'close' is Φ_K to being modular?

For each even integral lattice $\mathcal{L}$,
we define its usual theta function by:

$$\Theta_{\mathcal{L}}(\tau) := \sum_{n\in\mathcal{L}} e^{\pi i\tau||n||^2},$$

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It is a standard fact that the theta function $\Theta_{\mathcal{L}}(\tau)$ turns out to be a modular form, of weight $\frac{d}{2}$ and level N, where N divides $|\det(A)|$.

We define R to be the ring of all finite, rational linear combinations of theta functions $\Theta_{\mathcal{L}}$, for any d-dimensional even integral lattice $\mathcal{L}$, where we vary over all dimensions d.

Theorem (Folsom, Kohnen, R.)

If the polyhedral cone K is the Weyl chamber of a finite reflection group W, then the cone theta function $\Phi_{K,2\mathcal{L}_{root}}(\tau)$ is in the graded ring R.

The spirit of this result is that enough symmetry of the integer cone K will be reflected in some functional relations between the associated cone theta functions $\Phi_{K,\mathcal{L}_j}$, for various j-dimensional lattices $\mathcal{L}_j$ which lie on the boundaries of $K \cap \mathcal{L}$.

On the other hand, we have the following result, showing that conic theta functions are ‘usually’ very far from being in R.

Theorem (Folsom, Kohnen, R.)

Suppose that the d-dim'l polyhedral cone K has the solid angle ω_K at its vertex, located at the origin, and that $\mathbb{L} := A(\mathbb{Z}^d)$ is an even integral lattice of full rank.

If $\frac{\omega_K}{|det A|}$ is irrational, then $\Phi_{K,\mathcal{L}}(\tau)$ is not a modular form of weight k on any congruence subgroup, and for any $k \in \frac{1}{2}\mathbb{Z}$, $k \geq \frac{1}{2}$.

In the 2-dimensional case, we can classify the integer cones that have an irrational angle. As a consequence:

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Theorem (Folsom, Kohnen, R.)

Suppose we are given an integer cone $K \subset \mathbb{R}^2$, with *integer* edge vectors $w_1, w_2 \in \mathbb{Z}^2$.

Then $\Phi_{K,\mathbb{Z}^2}(\tau)$ is not a modular form of weight 1 for any congruence subgroup.

Open Problems

Problem 1. What are the necessary and sufficient conditions on the geometry of the cones K whose cone theta function belongs to the graded ring R?

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Problem 3. Which integer 3-dimensional cones have a rational spherical volume?

(This is closely related to the Cheeger-Simons rational simplex conjecture, so it is most likely quite challenging.)

Thank you