The class number one problem for curves of genus 2

Pınar Kılıçer

Universiteit Leiden - Université de Bordeaux

March 18, 2014

Proposition

There exist exactly 20 isomorphism classes of cyclic quartic CM-fields K for which $\mathbf{C}_{\Phi^r}$ is trivial.
There exist exactly 63 isomorphism classes of non-Galois quartic CM-fields K for which $\mathbf{C}_{\Phi^r}$ is trivial.

Proof

Non-Galois quartic case:
Let K be a non-Galois quartic CM-field of type Φ with intermediate field F and (K^r, Φ^r) be reflex of (K, Φ).

- If $\mathbf{C}_{\Phi^r}$ is trivial, then
  - (i) $F = \mathbb{Q}(\sqrt{p})$ and $F^r = \mathbb{Q}(\sqrt{q})$ where p and q are primes and $q \not\equiv 3 \pmod 4$,
  - (ii) $h_{K^r}/h_{F^r} = 2^{t-1}$ where t is the number of ramified primes in K^r/F^r,
  - (iii) $(p/q) = (q/p) = 1$ and the ramified primes $r_i \neq p$ in K^r/F^r satisfy $(r_i/p) = (r_i/q) = -1$.

Proof

- (Louboutin) Let N be the normal closure of K^r. If the discriminant $d_N \geq 222^8$, we have the lower bound

$$h_{K^r}/h_{F^r} \geq \frac{2}{\sqrt{e}\pi^2}\frac{\sqrt{d_{K^r}/d_{F^r}}}{(\log(d_{K^r}/d_{F^r}) + 5\kappa/4)^2},$$

  where $\kappa = 0.046191417\cdots$.
  - For example, if $p \equiv q \equiv 1 \pmod 4$ and 2 does not ramify in K^r. Then by (iii), $d_{K^r}/d_{F^r} = pqr_1^2 \cdots r_{t-1}^2$.
  - For $t \geq 0$, let Δ_t denote the product of the first t odd primes. Then $d_{K^r}/d_{F^r} \geq pq(\Delta_{t-1})^2$.
  - Divide the both sides of the inequality by 2^{t-1}. Then the LHS is 1 and the RHS goes to ∞ as $t \to \infty$.
  - In fact $t < 8$ and hence $d_{K^r}/d_{F^r} \leq 12 \cdot 10^{10}$.

Proof

- SAGE:
  - Construct all non-Galois quartic CM-fields K^r with either
    - $d_N \leq 222^8$ or
    - $d_{K^r}/d_{F^r} \leq 12 \cdot 10^{10}$ and $t < 8$

    that contain the real quadratic subfield $F^r = \mathbb{Q}(\sqrt{q})$ and have $F = \mathbb{Q}(\sqrt{p})$ as the real quadratic subfield of K such that
    - p and q are prime numbers with $q \not\equiv 3 \pmod 4$ and $(p/q) = (q/p) = 1$,
    - $(r_i/p) = (r_i/q) = -1$ where r_i's are prime numbers that ramify in K^r/F^r.
  - Eliminate fields K by finding totally split primes under the bound $\frac{\sqrt{d_K/d_F^2}}{4}$. (Most of the fields are eliminated in this step.)
  - Eliminate fields K^r that have prime ideals in $I_{K^r} \setminus H_{\Phi^r}$ with norm below the bound $12 \cdot \log(|d_{K^r}|)^2$.
  - Compute class groups and check whether $\mathbf{C}_{\Phi^r}$ is trivial.

Further Directions

- Genus-2 CM curves for which $J(C)$ is isogenous to $E_1 \times E_2$,
- CM Picard curves: $y^3 = f(x)$ with $\deg(f) = 4$,
- CM hyperelliptic curves of genus 3,
- arbitrary CM curves of genus 3.