Class polynomials for abelian surfaces

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(joint work with Emmanuel Thomé)

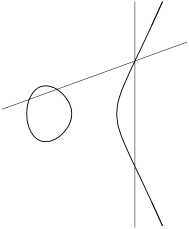
CM2

1. Elliptic curves
   - Applications of elliptic curves
   - Complex multiplication theory
   - Algorithms

2. Abelian surfaces
   - Complex multiplication theory
   - Algorithms
   - Implementation and examples

Elliptic curves

- $E : Y^2 = X^3 + aX + b, \quad a, b \in \mathbb{F}_p$
- Abelian variety of dimension 1 $\Rightarrow$ finite group



- Hasse 1934

$$|\#E(\mathbb{F}_p) - (p+1)| \leqslant 2\sqrt{p}$$

- Moduli space of dimension 1 parameterised by invariant

$$j = 1728 \frac{4a^3}{4a^3 + 27b^2}$$

Primality proofs (ECPP)

If $P \in E(\mathbb{Z}/N_1\mathbb{Z})$ with P of prime order N_2,

$$N_2 > \left(\sqrt[4]{N_1} + 1\right)^2,$$

then N_1 is prime.

Record: 25 050 decimal digits (Morain 2010)

Cryptography

- Discrete logarithm based cryptography
  - Need prime cardinality
  - Prefer random curves
- Pairing-based cryptography Weil and (reduced) Tate pairing

$$e : E(\mathbb{F}_p)[\ell] \times E(\mathbb{F}_{p^k})[\ell] \to \mathbb{F}_{p^k}^\times[\ell]$$

  - Bilinear: $e(aP, bQ) = e(P, Q)^{ab}$
  - An exponential number of cryptographic primitives...
  - Need CM constructions for suitable curves.

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Complex multiplication

Deuring 1941: The endomorphism ring of an (ordinary) elliptic curve is either $\mathbb{Z}$, or an order

$$\mathcal{O}_D = \left[1, \frac{D + \sqrt{D}}{2} \right]_{\mathbb{Z}}$$

of discriminant $D < 0$ in $K = \mathbb{Q}(\sqrt{D})$.

$\boxed{E \text{ with complex multiplication by } \mathcal{O}_D \text{ / by } D}$

- Over $\mathbb{C}$: usually $\mathbb{Z}$, sometimes $\mathcal{O}_D$
- Over $\mathbb{F}_p$: always $\mathcal{O}_D$!

Complex multiplication

- Frobenius: $\pi : (x, y) \mapsto (x^p, y^p)$, fixes $E(\mathbb{F}_p)$
- Deuring 1941: Any (ordinary) curve over $\mathbb{F}_p$ is the reduction of a curve over $\mathbb{C}$ with the same endomorphism ring.
- Hasse: $\pi = \frac{t+v\sqrt{D}}{2}$, $\mathrm{Tr}(\pi) = t$, $\mathrm{N}(\pi) = \frac{t^2-v^2D}{4} = p$

$$\#E(\mathbb{F}_p) = p + 1 - t$$

Effective complex multiplication

Given D, what are the curves over $\mathbb{C}$ with CM by D?

- Modular invariant

$$j : \mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\} \to \mathbb{C}$$

- $\varphi : K = \mathbb{Q}(\sqrt{D}) \to \mathbb{C}$ embedding
- $\mathfrak{a} = (\alpha_1, \alpha_2)$ ideal of $\mathcal{O}_D$ with basis quotient $\tau = \varphi\left(\frac{\alpha_2}{\alpha_1}\right) \in \mathbb{H}$
- $j(\tau)$ depends only on the ideal class of $\mathfrak{a}$;
  determines the $h = \#\mathrm{Cl}(\mathcal{O}_D)$ curves with CM by D.

First main theorem of complex multiplication

$$\begin{array}{c} \Omega_D = K(j(\mathfrak{a})) \\ | \\ K = \mathbb{Q}(\sqrt{D}) \\ | \\ \mathbb{Q} \end{array}$$

$$\begin{aligned} \Omega_D &= \text{Hilbert class field of } K \text{ (for } D \text{ fundamental discriminant)} \\ &= \text{maximal abelian, unramified extension of } K \end{aligned}$$

$$\sigma : \mathrm{Cl}(\mathcal{O}_D) \xrightarrow{\simeq} \mathsf{Gal}(\Omega_D/K)$$

$$j(\mathfrak{a})^{\sigma(\mathfrak{b})} = j(\mathfrak{a}\mathfrak{b}^{-1})$$

Main algorithm

- Fix $D < 0$ and p prime s.t. $p = \frac{t^2 - v^2 D}{4}$ and $N = p + 1 - t$ convenient
- Enumerate the h ideal classes of $\mathcal{O}_D$:

$$\left(A_i, \frac{-B_i + \sqrt{D}}{2} \right)$$

- Compute over $\mathbb{C}$ the class polynomial

$$H(X) = \prod_{i=1}^{h} \left(X - j \left(\frac{-B_i + \sqrt{D}}{2A_i} \right) \right) \in \mathbb{Z}[X]$$

- Find a root $\bar{j}$ modulo p
- Write down the curve $E : Y^2 = X^3 + aX + b$ with

$$c = \frac{\bar{j}}{1728 - \bar{j}}, \quad a = 3c, \quad b = 2c$$

Complexity

- Size of H
  - Degree $h \in \tilde{O}\left(\sqrt{|D|}\right)$ (Littlewood 1928)
  - Coefficients with $\tilde{O}\left(\sqrt{|D|}\right)$ digits (Schoof 1991, E. 2009)
  - Total size $\tilde{O}(|D|)$
- Evaluation of j: $\tilde{O}\left(\sqrt{|D|}\right)$
  - Precision: $\tilde{O}\left(\sqrt{|D|}\right)$ digits
  - Multievaluation of the "polynomial" j (E. 2009)
  - Arithmetic-geometric mean (Dupont 2006)
- Total complexity (E. 2009)

$\boxed{\tilde{O}(|D|) \text{ — quasi-linear in the output size!}}$

Implementation

- Record (E. 2009) (with class invariants)
  - $D = -2\,093\,236\,031$
  - $h = 100\,000$
  - Precision $264\,727$ bits
  - $260\,000$ s $= 3$ d CPU time
  - 5 GB
- Software
  - GNU MPC: complex floating point arithmetic in arbitrary precision with guaranteed rounding
    - Based on MPFR and GMP
    - LGPL
  - MPFRCX: polynomials with real (MPFR) and complex (MPC) coefficients
    - LGPL
  - `cm`: class polynomials and CM curves
    - GPL

http://www.multiprecision.org/

Further algorithms

- *p*-adic lift
  - Couveignes–Henocq 2002, Bröker 2006
- Chinese remaindering
  - Enumerate CM curves over $\mathbb{F}_p$, compute $H \bmod p$
  - Lift to $\mathbb{Z}$ or directly to $\mathbb{Z}/P\mathbb{Z}$
  - Belding–Bröker–E.–Lauter 2008 following an idea by D. Bernstein, Sutherland 2009, E.–Sutherland 2010
- Record (E.–Sutherland 2010)
  - $D = -1\,000\,000\,013\,079\,299$
  - $h = 10\,034, 174$
  - $P \approx 2^{254}$
  - Precision $21\,533\,832$ bits
  - $438\,709$ primes of $\leqslant 53$ bits
  - 200 d CPU time
  - Size $\bmod P \approx 200$ MB
  - Size over $\mathbb{Z} \approx 2$ PB

AGM

Dupont 2006: One can evaluate j at precision n in time

$$O(\log n\, M(n)) = \tilde{O}(n).$$

Idea of the algorithm:
Newton iterations on a function built with the arithmetic-geometric mean (AGM)

Theta constants — definition

$$a, b \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}; \qquad q = e^{\pi i \tau}$$

$$\vartheta_{a,b}(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i((n+a)\tau(n+a)+2(n+a)b)} = e^{2\pi i a b} \sum_{n \in \mathbb{Z}} (e^{2\pi i b})^n q^{(n+a)^2}$$

$$\begin{aligned}
\vartheta_{0,0}(\tau) &= \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \dots \\
\vartheta_{0,\frac{1}{2}}(\tau) &= \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} = 1 - 2q + 2q^4 - 2q^9 + \dots \\
\vartheta_{\frac{1}{2},0}(\tau) &= \sum_{n \in \mathbb{Z}} q^{(2n+1)^2/4} = q^{1/4}\left(1 + 2q + 2q^3 + \dots\right) \\
\vartheta_{\frac{1}{2},\frac{1}{2}}(\mathbb{Z}) &= 0
\end{aligned}$$

Theta constants — duplication formulæ

$$
\begin{aligned}
\vartheta_{0,0}^2(2\tau) &= \frac{\vartheta_{0,0}^2(\tau) + \vartheta_{0,\frac{1}{2}}^2(\tau)}{2} \\
\vartheta_{0,\frac{1}{2}}^2(2\tau) &= \sqrt{\vartheta_{0,0}^2(\tau)\vartheta_{0,\frac{1}{2}}^2(\tau)}
\end{aligned}
$$

AGM

$$\vartheta_{0,0}^2(2\tau) = \frac{\vartheta_{0,0}^2(\tau) + \vartheta_{0,\frac{1}{2}}^2(\tau)}{2}$$

$$\vartheta_{0,\frac{1}{2}}^2(2\tau) = \sqrt{\vartheta_{0,0}^2(\tau)\vartheta_{0,\frac{1}{2}}^2(\tau)}$$

AGM for $a,\ b \in \mathbb{C}$

- $a_0 = a,\ b_0 = b$
- $a_{n+1} = \frac{a_n + b_n}{2}$
- $b_{n+1} = \sqrt{a_n b_n}$
- converges quadratically towards a common limit $\mathrm{AGM}(a, b)$

$\boxed{\text{Evaluated in time } O(\log n\, M(n)) \text{ at precision } n.}$

Theta quotients

$$\mathrm{AGM}(a, b) = a \cdot \mathrm{AGM}(1, b/a) =: a \cdot M(b/a)$$

- $k'(z) = \left(\frac{\vartheta_{0,\frac{1}{2}}(z)}{\vartheta_{0,0}(z)} \right)^2$
- $k(z) = \left(\frac{\vartheta_{\frac{1}{2},0}(z)}{\vartheta_{0,0}(z)} \right)^2$
- $k^2(z) + k'^2(z) = 1$
- $j = 256 \frac{(1-k'^2+k'^4)^3}{k'^4(1-k'^2)^2}$

Newton iterations

- $M(k'(\tau)) = \frac{1}{\vartheta_{0,0}^2(\tau)}$
- $M(k(\tau)) = M(k'(S\tau)) = \frac{1}{\vartheta_{0,0}^2(S\tau)} = \frac{i}{\tau\vartheta_{0,0}^2(\tau)}$
- $k^2(\tau) + k'^2(\tau) = 1$
- $f_\tau(x) = iM(x) - \tau M(\sqrt{1-x^2})$
- $f_\tau(k'(\tau)) = 0$

$$x_{n+1} \leftarrow x_n - \frac{f_\tau(x_n)}{f'_\tau(x_n)}$$

converges quadratically towards $k'(\tau)$

Evaluated in time $O(\log n\, M(n))$ at precision n

Genus 2 curves and ppav of dimension 2

- $\mathcal{C} : Y^2 = X^5 + aX^3 + bX^2 + cX + d$ hyperelliptic curve of genus 2
- Jacobian is a principally polarised abelian surface (ppas)
- Moduli space of dimension 3
  parameterised by Igusa invariants i_1, i_2, i_3
- Frobenius endomorphism gives cardinality of Jacobian over $\mathbb{F}_p$
  $\Rightarrow$ source of cryptographic curves

Endomorphism rings and period matrices

- End $= \mathcal{O} \subseteq K = \mathbb{Q}[X]/(X^4 + AX^2 + B)$ with $D = A^2 - 4B > 0$
  CM field of degree 4

$$\begin{array}{c} K = K_0\left(\pm\sqrt{\frac{-A\pm\sqrt{D}}{2}}\right) \\ | \\ K_0 = \mathbb{Q}(\sqrt{D}) \\ | \\ \mathbb{Q} \end{array}$$

- CM types $\Phi = (\varphi_1, \varphi_2)$, $\Phi' = (\varphi_1, \overline{\varphi}_2)$, embeddings: $K \to \mathbb{C}$
- $(\mathfrak{a}, \xi)$ s.t. $(\mathfrak{a}\overline{\mathfrak{a}}\mathcal{D}_{K/\mathbb{Q}})^{-1} = (\xi)$, $\varphi_1(\xi)$, $\varphi_2(\xi) \in i\mathbb{R}_{>0}$ (polarisation)
- $(\mathfrak{a}, \xi) \rightsquigarrow \tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}$ with $\Im(\tau)$ positive definite (period matrix)

Theta constants

$$a, b \in \left(\frac{1}{2}\mathbb{Z}/\mathbb{Z}\right)^2$$

$$\vartheta_{a,b}(\tau) = \sum_{n \in \mathbb{Z}^2} e^{\pi i \left((n+a)^T \tau (n+a) + 2(n+a)^T b\right)}$$

10 non-zero theta constants

Siegel modular forms

$$I_4 = \sum_{10\, i} \vartheta_i^8$$

$$I_6 = \sum_{\text{certain } 60\, i,j,k} \pm(\vartheta_i \vartheta_j \vartheta_k)^4$$

$$I_{10} = \prod_{10 i} \vartheta_i^2$$

$$I_{12} = \sum_{15} \prod_{6\, i} \vartheta_i^4$$

Igusa invariants

according to Streng 2010

$$i_1 = \frac{I_4 I_6}{I_{10}}$$

$$i_2 = \frac{I_{12} I_4^2}{I_{10}^2}$$

$$i_3 = \frac{I_4^5}{I_{10}^2}$$

Class fields (dihedral case)

$$
\begin{tikzcd}[arrows=dash]
 & & & \Omega \arrow[d] \\
 & & & K^r(i_1(\tau)) = K^r(i_2(\tau)) = K^r(i_1(\tau)) \arrow[ddl] \\
 & L \arrow[dl] \arrow[dr] & & \\
K \arrow[d] & & K^r \arrow[d] & \\
K_0 \arrow[dr] & & K_0^r \arrow[dl] & \\
 & \mathbb{Q} & &
\end{tikzcd}
$$

CM2

Algorithms

- Complex analytic
  - Spallek 1994
  - Weng 2001
  - Streng 2010
- p-adic lift
  - Gaudry–Houtmann–Kohel–Ritzenthaler–Weng 2006
- Chinese remaindering
  - Eisenträger–Lauter 2005
  - Lauter–Robert 2012
- Our contributions to the complex-analytic algorithm
  - Quasi-linear evaluation of theta constants (following Dupont 2006)
    $\Rightarrow$ quasi-linear computation of class polynomials (Streng 2010)
    $\Rightarrow$ most efficient algorithm
  - Direct computation of irreducible factors, over K_0^r instead of $\mathbb{Q}$ (following Streng 2010)
  - Software

Main algorithm (dihedral case)

- Let $h_0 = \#\mathrm{Cl}(K_0)$, $h_1 = \#\mathrm{Cl}(K)/h_0$
- Consider the two CM-types Φ and Φ', enumerate $\mathrm{Cl}(K)$
- Compute

$$S(K, \Phi) = \left\{(\mathfrak{a}, \xi) : (\mathfrak{a}\bar{\mathfrak{a}}\mathcal{D}_{K/\mathbb{Q}})^{-1} = (\xi), \Phi(\xi) \in (i\mathbb{R}_{>0})^2\right\} / \sim$$

and $S(K, \Phi')$, where

$$(\mathfrak{a}, \xi) \sim (x\mathfrak{a}, (x\bar{x})^{-1}\xi)$$

- $\#S(K, \Phi) = \#S(K, \Phi') = h_1 \Rightarrow$ period matrices τ_i, τ_i'
- Evaluate the $\vartheta_{a,b}(\tau_i^{(')})$ and deduce the $i_k(\tau_i^{(')})$

Main algorithm (dihedral case)

- Compute the first class polynomial

$$H_1(X) = \prod_{i=1}^{h_1} (X - i_1(\tau_i)) \prod_{i=1}^{h_1} (X - i_1(\tau_i')) \in \mathbb{Q}[X]$$

- Compute the Hecke representations of the algebraic numbers $i_k(\tau_i)$ with respect to H_1:

$$\hat{H}_k(X) = \text{ polynomial of degree } h_1 - 1 \text{ such that } i_k(\tau_i) = \frac{\hat{H}_k(i_1(\tau_i))}{H_1'(i_1(\tau_i))}$$

(roughly Lagrange interpolation)

Borchardt sequences

$$
\begin{aligned}
a_{n+1} &= \frac{a_n + b_n + c_n + d_n}{4} \\
b_{n+1} &= \frac{\sqrt{a_n}\sqrt{b_n} + \sqrt{c_n}\sqrt{d_n}}{2} \\
c_{n+1} &= \frac{\sqrt{a_n}\sqrt{c_n} + \sqrt{b_n}\sqrt{d_n}}{2} \\
d_{n+1} &= \frac{\sqrt{a_n}\sqrt{d_n} + \sqrt{b_n}\sqrt{c_n}}{2}
\end{aligned}
$$

Common limit: Borchardt mean $B_2(a_0, b_0, c_0, d_0)$
Related to duplication formulæ of four fundamental theta constants $\vartheta_0, \ldots, \vartheta_3$.

τ from $(\vartheta_j(\tau/2)/\vartheta_0(\tau/2))_{j=1,2,3}$

- Compute $(\vartheta_j^2(\tau)/\vartheta_0^2(\tau/2))_{j=0,1,2,3,4,6,8,9,12,15}$ (duplication)
- Compute $B_2((\vartheta_j^2(\tau)/\vartheta_0^2(\tau/2))_{j=0,1,2,3} = \frac{1}{\vartheta_0^2(\tau/2)}$
- Compute $(\vartheta_j^2(\tau))_{j=0,1,2,3,4,6,8,9,12,15}$
- Compute

$$\begin{aligned} u_1 &= B_2((\vartheta_j^2(\tau))_{j=4,0,6,2} \\ u_3 &= B_2((\vartheta_j^2(\tau))_{j=8,9,0,1} \\ u_2 &= B_2((\vartheta_j^2(\tau))_{j=0,8,4,12} \end{aligned}$$

- Return

$$\tau_1 = \frac{i}{u_1}, \tau_3 = \frac{i}{u_3}, \tau_2 = \pm\sqrt{\frac{1}{u_2} + \tau_1\tau_3}$$

Improvements to Dupont 2006

- Streamline the computations
- Replace

$$\frac{\partial f}{\partial \tau_i}(\tau)$$

by

$$\frac{f(\tau + \varepsilon e_i) - f(\tau)}{\varepsilon}$$

(gain about 25%)

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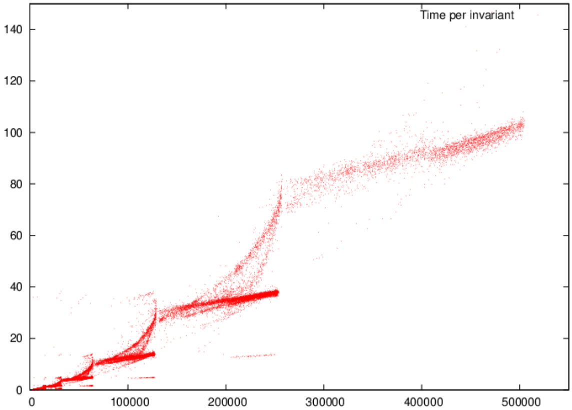
Implementation

- Number theoretic computations: $\mathfrak{C}(K)$, (reduced) period matrices
  - Pari/GP
  - negligible effort
- Evaluation of theta and invariants
  - C
  - Libraries: GMP, MPFR, MPC
  - MPI for parallelisation
- Polynomial operations
  - MPFRCX
  - MPI for (partial) parallelisation

`http://cmh.gforge.inria.fr/`

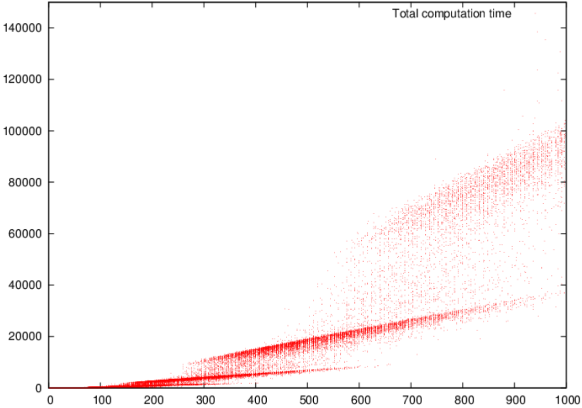
Quasi-linear complexity

- required precision = coefficient size
- time per invariant = $\tilde{O}$(precision)



Quasi-linear complexity

- required precision $=$ coefficient size
- time per invariant $= \tilde{O}$ (precision)
- total time $= \tilde{O}$ (output size)



Record example

- K defined by $X^4 + 1357X^2 + 2122$, $D = 1832961$, $h_0 = 8$
- $\mathfrak{C}(K) \simeq \mathbb{Z}/4402\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- PARI/GP: 4 min (reduction of period matrices)
- Precision: $7\,536\,929$ bits
- Invariants:
  - Last Newton lift: ≈ 3000 s per invariant (≈ 1200 second-to-last)
  - ≈ 2 d wallclock time on 160 processors
- Polynomial operations (partially parallelised):
  - ≈ 1 d wallclock time (40 processors, 1 TB memory)
- Algebraic coefficient recognition:
  - ≈ 2600 s per coefficient
  - ≈ 10 d wallclock time on 160 processors
- Size: 56 GB
- # primes in denominator: 3465
- Largest prime in denominator: $242\,363\,767$

Bound: $54\,004\,867\,207\,824$

Conclusion

- Quasi-linear algorithm for class polynomials in dimension 2
- Computation of invariants
  - efficient
  - arbitrarily parallel
- As can be expected: Memory becomes the bottleneck
- Better parallelisation/distribution of polynomial operations required
- Quasi-linear LLL in dimension 3 desirable
- Next steps:
  better understand the denominators
  smaller class invariants (work in progress with M. Streng)

`http://cmh.gforge.inria.fr/`

`http://hal.archives-ouvertes.fr/hal-00823745/`