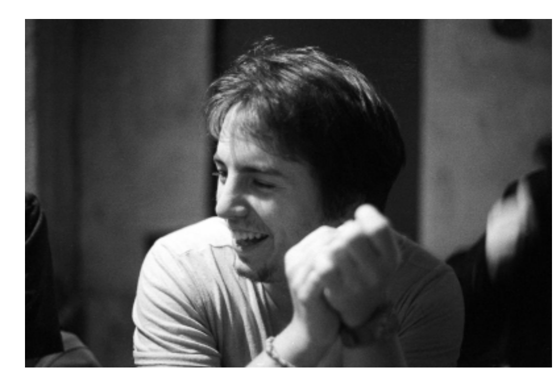
Exact Semidefinite Programming Bounds for Packing Problems

Maria Dostert (EPFL)

joint work with

David de Laat
(TU Delft)

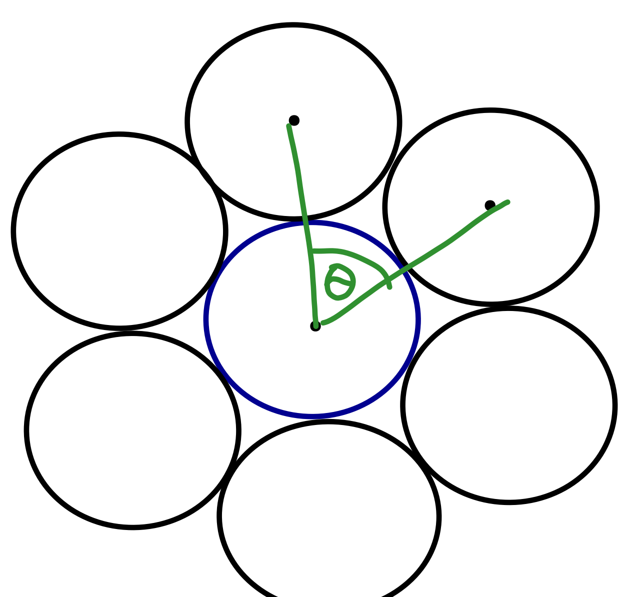


Philippe Moustrou
(UiT-The Arctic University of Norway)

Kissing number of spheres

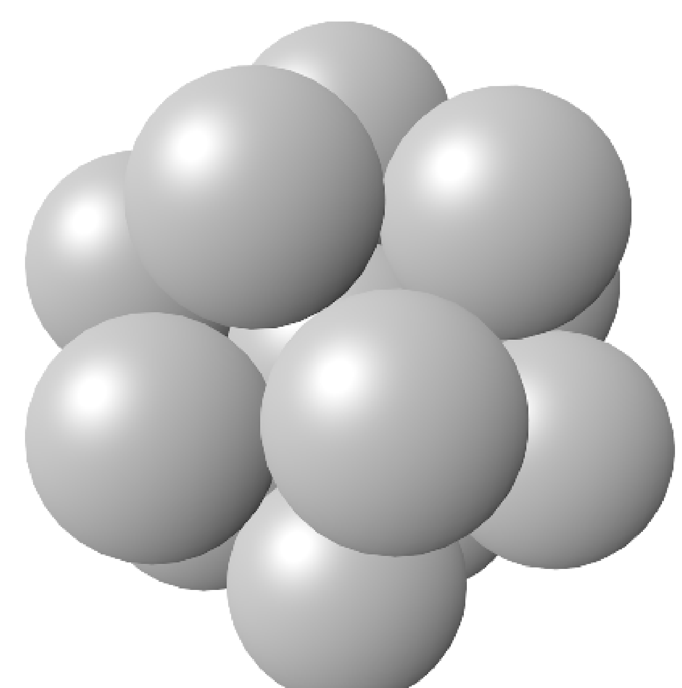
Kissing number of sphere = max number of pairwise non-overlapping unit spheres that can touch simultaneously a central sphere

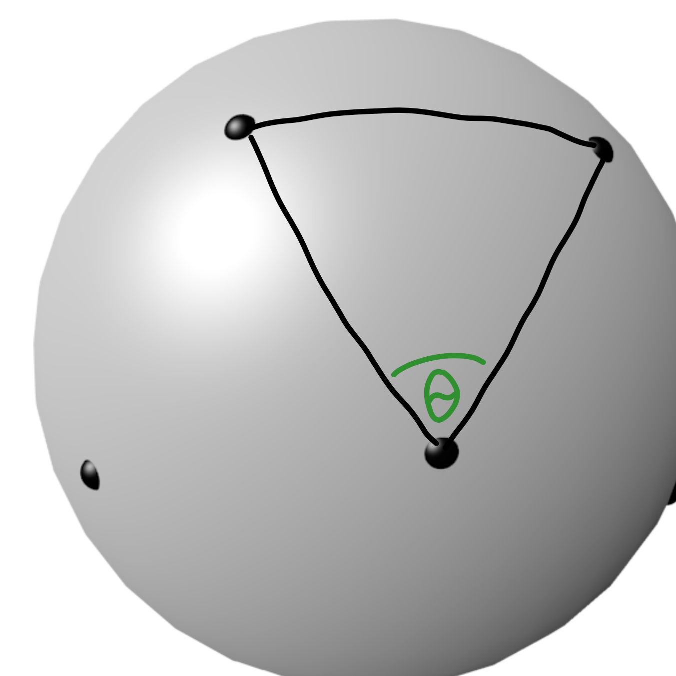
Dimension 2: kissing number is 6



$\theta = \pi/3$

Dimension 3: kissing number is 12 (Schütte, van der Waerden, 1953)





Kissing number of hemisphere in dimension 8

Kissing number of hemisphere = max number of pairwise non-overlapping unit spheres that can touch simultaneously a central hemisphere

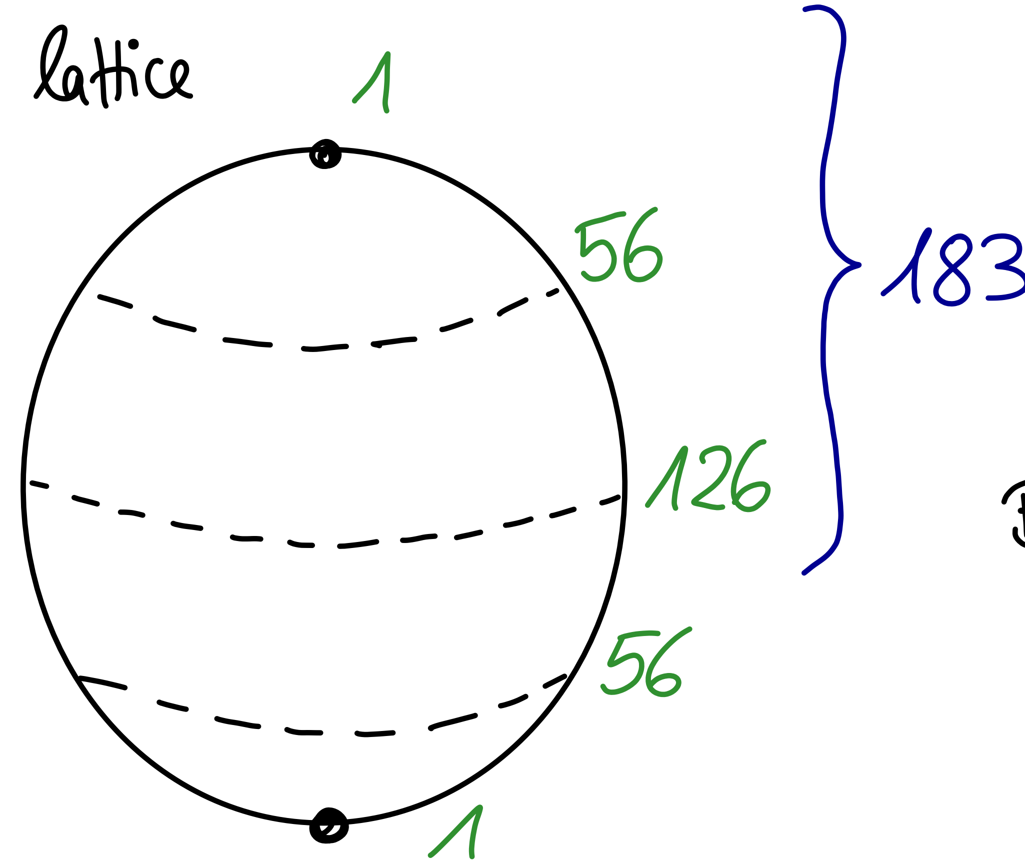
kissing number of sphere in dim 8 = 240

configuration given by E_8 lattice

(Odlyzko, Sloane 1979, Levenshtein 1979)

unique up to isometry

(Bannai, Sloane 1981)



$\Rightarrow E_8$ gives a kissing configuration of 183 points

Bachoc, Vallentin (2007):

upper bound 183.012 $\Rightarrow$ optimum 183

Spherical code

Euclidean space $\mathbb{R}^n$, inner product $x \cdot y = \sum_{i=1}^{n} x_i y_i$

$$A(n, \Theta) = \max\{ |C| : C \subset S^{n-1} \text{ with } c \cdot c' \leq \cos\Theta \text{ for } c, c' \in C,\ c \neq c' \}$$

Finding $A(n, \pi/3)$ is equivalent to the kissing problem

Spherical code

Euclidean space $\mathbb{R}^n$, inner product $x \cdot y = \sum_{i=1}^{n} x_i y_i$

$$A(n,\theta) = \max\{|C| : C \subset S^{n-1} \text{ with } c \cdot c' \leq \cos\theta \text{ for } c, c' \in C,\ c \neq c'\}$$

Finding $A(n, \pi/3)$ is equivalent to the kissing problem

Let $\mathrm{Cap}(e,\phi) = \{x \in S^{n-1} : e \cdot x \geq \cos\phi\}$

$$A(n,\theta,\phi) = \max\{|C| : C \subset \mathrm{Cap}(e,\phi) \text{ with } c \cdot c' \leq \cos\theta \text{ for } c, c' \in C,\ c \neq c'\}$$

Semidefinite program

Let

$$\Delta = \{(u,v,t) : \cos\phi \le u \le v \le 1,\ -1 \le t \le \cos\theta,\ 1+2uvt-u^2-v^2-t^2 \ge 0\}$$

and

$$\Delta_0 = \{(u,u,1) : \cos\phi \le u \le 1\}$$

Theorem (Bachoc, Vallentin 2007)

$$A(n,\theta,\phi) \le \min\{1+M :$$

$$F_k \succeq 0 \quad \text{for all } k=0,\dots,d,$$

$$\text{(i)} \sum_{k=1}^{d} \langle F_k, \overline{Y}_k^n(u,u,1)\rangle \le M \quad \text{for all } (u,u,1) \in \Delta_0,$$

$$\text{(ii)} \sum_{k=1}^{d} \langle F_k, \overline{Y}_k^n(u,v,t)\rangle \le -1 \quad \text{for all } (u,v,t) \in \Delta\}$$

Complementary Slackness

Let $C \subset \mathrm{Cap}(e,\phi) = \{x \in S^{n-1} : e \cdot x \geq \cos\phi\}$ with $c \cdot c' \leq \cos\Theta$ for $c, c' \in C$, $c \neq c'$

Let F_k be feasible and $F(u,v,t) := \sum_{k=1}^{d} \langle F_k, \overline{Y}_k^n(u,v,t)\rangle$

Complementary Slackness

Let $C \subset \mathrm{Cap}(e,\phi) = \{x \in S^{n-1} : e \cdot x \geq \cos\phi\}$ with $c \cdot c' \leq \cos\Theta$ for $c, c' \in C$, $c \neq c'$

Let F_k be feasible and $F(u,v,t) := \sum_{k=1}^{d} \langle F_k, \overline{Y}_k^n(u,v,t) \rangle$

$$\sum_{(c,c') \in C^2} F(e\cdot c, e\cdot c', c\cdot c') = \sum_{\substack{(c,c') \in C^2 \\ c \neq c'}} \underbrace{F(e\cdot c, e\cdot c', c\cdot c')}_{\leq -1 \text{ (ii)}} + \sum_{c \in C} \underbrace{F(e\cdot c, e\cdot c, 1)}_{\leq M \text{ (i)}} \leq |C| \left(-|C| + 1 + M\right)$$

Complementary Slackness

Let $C \subset \mathrm{Cap}(e,\phi) = \{x \in S^{n-1} : e \cdot x \geq \cos\phi\}$ with $c \cdot c' \leq \cos\Theta$ for $c, c' \in C$, $c \neq c'$

Let F_k be feasible and $F(u,v,t) := \sum_{k=1}^{d} \langle F_k, \overline{Y}_k^n(u,v,t) \rangle$

$$\sum_{(c,c') \in C^2} F(e\cdot c, e\cdot c', c\cdot c') = \sum_{\substack{(c,c') \in C^2 \\ c \neq c'}} \underbrace{F(e\cdot c, e\cdot c', c\cdot c')}_{\leq -1 \text{ (ii)}} + \sum_{c \in C} \underbrace{F(e\cdot c, e\cdot c, 1)}_{\leq M \text{ (i)}} \leq |C| \left(-|C| + 1 + M\right)$$

$$\sum_{(c,c') \in C^2} F(e\cdot c, e\cdot c', c\cdot c') = \sum_{k=1}^{d} \Big\langle \underset{\succeq 0}{F_k}, \underbrace{\sum_{(c,c') \in C^2} \overline{Y}_k^n(e\cdot c, e\cdot c', c\cdot c')}_{\succeq 0} \Big\rangle \geq 0$$

Complementary Slackness

Let $C \subset \mathrm{Cap}(e,\phi) = \{x \in S^{n-1} : e\cdot x \geq \cos\phi\}$ with $c\cdot c' \leq \cos\theta$ for $c, c' \in C$, $c \neq c'$

Let F_k be feasible and $F(u,v,t) := \sum_{k=1}^{d} \langle F_k, \bar{Y}_k^n(u,v,t)\rangle$

$$\sum_{(c,c')\in C^2} F(e\cdot c, e\cdot c', c\cdot c') = \sum_{\substack{(c,c')\in C^2 \\ c\neq c'}} \underbrace{F(e\cdot c, e\cdot c', c\cdot c')}_{\leq -1\ (ii)} + \sum_{c\in C} \underbrace{F(e\cdot c, e\cdot c, 1)}_{\leq M\ (i)} \leq |C|\,(-|C|+1+M)$$

$$\sum_{(c,c')\in C^2} F(e\cdot c, e\cdot c', c\cdot c') = \sum_{k=1}^{d} \langle \underset{\succeq 0}{F_k}, \underbrace{\sum_{(c,c')\in C^2} \bar{Y}_k^n(e\cdot c, e\cdot c', c\cdot c')}_{\succeq 0} \rangle \geq 0$$

$\Rightarrow 0 \leq |C|(-|C|+1+M)$ $\Rightarrow |C| \leq 1+M$

Rational Solution ?

$$\text{SDP:}\quad \min\left\{\sum_{i=1}^{n}\langle C_i, X_i\rangle : \sum_{i=1}^{n}\langle A_{j_i}, X_i\rangle = b_j \text{ for } j=1,\dots,m,\ X_i \succeq 0 \text{ for } i=1,\dots,n\right\}$$

Solver returns floating point solution $X_1,\dots,X_n$

Rational Solution?

$$\text{SDP:} \quad \min\left\{ \sum_{i=1}^{n} \langle C_i, X_i \rangle : \sum_{i=1}^{n} \langle A_{j_i}, X_i \rangle = b_j \text{ for } j=1,\dots,m, \; X_i \succeq 0 \text{ for } i=1,\dots,n \right\}$$

Solver returns floating point solution $X_1, \dots, X_n$

Rounding to rational solution such that

1) linear system satisfied

2) all rational matrices positive semidefinite

$\boxed{\text{Rational Solution?}}$

SDP: $\min \left\{ \sum_{i=1}^{n} \langle C_i, X_i \rangle : \sum_{i=1}^{n} \langle A_{j,i}, X_i \rangle = b_j \text{ for } j=1,\dots,m, \; X_i \succeq 0 \text{ for } i=1,\dots,n \right\}$

Solver returns floating point solution $X_1, \dots, X_n$

Rounding to rational solution such that

1) linear system satisfied
2) all rational matrices positive semidefinite

Determine good floating point solution:

use high precision solver (SDPA GMP)

Rounding in the affine space

If $A_1, \dots, A_n$ rational, we can rationalize $X_1, \dots, X_n$ s.t. all linear constraints are satisfied exactly

Rounding in the affine space

If $A_1, \dots, A_n$ rational, we can rationalize $X_1, \dots, X_n$ s.t. all linear constraints are satisfied exactly

* Reduce variables using u-v symmetries: $110376 \Rightarrow 37651$

Rounding in the affine space

If $A_1, \dots, A_n$ rational, we can rationalize $X_1, \dots, X_n$ s.t. all linear constraints are satisfied exactly

* Reduce variables using u-v symmetries: $110376 \Rightarrow 37651$
* Add constraints (Complementary Slackness & derivative)

Rounding in the affine space

If $A_1, \dots, A_n$ rational, we can rationalize $X_1, \dots, X_n$ s.t. all linear constraints are satisfied exactly

* Reduce variables using u-v symmetries: $110376 \Rightarrow 37651$
* Add constraints (Complementary Slackness & derivative)
* Reduce linear system to reduced row echelon in rational arithmetic

Rounding in the affine space

If $A_1, \dots, A_n$ rational, we can rationalize $X_1, \dots, X_n$ s.t. all linear constraints are satisfied exactly

* Reduce variables using u-v symmetries: $110376 \Rightarrow 37651$
* Add constraints (Complementary Slackness & derivative)
* Reduce linear system to reduced row echelon in rational arithmetic
* For each variable add constraint that forces this variable to be close to corresponding floating point value

Rounding in the affine space

If $A_1, \dots, A_n$ rational, we can rationalize $X_1, \dots, X_n$ s.t. all linear constraints are satisfied exactly

* Reduce variables using u-v symmetries: $110376 \Rightarrow 37651$
* Add constraints (Complementary Slackness & derivative)
* Reduce linear system to reduced row echelon in rational arithmetic
* For each variable add constraint that forces this variable to be close to corresponding floating point value

Backsubstitution $\Rightarrow \tilde{X}_1, \dots, \tilde{X}_n$ rational, satisfying lin. system

Rounding in the affine space

If $A_1, \dots, A_n$ rational, we can rationalize $X_1, \dots, X_n$ s.t. all linear constraints are satisfied exactly

- Reduce variables using u-v symmetries: $110376 \Rightarrow 37651$
- Add constraints (Complementary Slackness & derivative)
- Reduce linear system to reduced row echelon in rational arithmetic
- For each variable add constraint that forces this variable to be close to corresponding floating point value

Backsubstitution $\Rightarrow \tilde{X}_1, \dots, \tilde{X}_n$ rational, satisfying lin. system

Problem: System very large $\Rightarrow$ use HNF algorithm (Fieker, Hofman, Sircena) in Julia computational number theory package Hecke

Positive Semidefinite Constraints

Problem: Even if $X_1, \dots, X_n$ psd, $\tilde{X}_1, \dots, \tilde{X}_n$ often not psd!

Positive Semidefinite Constraints

Problem: Even if $X_1, \ldots, X_n$ psd, $\tilde{X}_1, \ldots, \tilde{X}_n$ often not psd!

* Solve SDP as decision problem

  $\Rightarrow X_1, \ldots, X_n$ (approx.) in the relative interior of opt. face

Positive Semidefinite Constraints

Problem: Even if $X_1, \dots, X_n$ psd, $\tilde{X}_1, \dots, \tilde{X}_n$ often not psd!

* Solve SDP as decision problem

  $\Rightarrow X_1, \dots, X_n$ (approx.) in the relative interior of opt. face

* Note: interior of psd cone is cone of pos. definite matrices

* Remaining near zero eigenvalues are probably there for a (structural) reason

Positive Semidefinite Constraints

Problem: Even if $X_1, \dots, X_n$ psd, $\tilde{X}_1, \dots, \tilde{X}_n$ often not psd!

* Solve SDP as decision problem
    $\Rightarrow X_1, \dots, X_n$ (approx.) in the relative interior of opt. face
* Note: interior of psd cone is cone of pos. definite matrices
* Remaining near zero eigenvalues are probably there for a (structural) reason

$\Rightarrow$ We expect there to be rational bases of the kernel with small denominators

Positive Semidefinite Constraints

Problem: Even if $X_1, \dots, X_n$ psd, $\tilde{X}_1, \dots, \tilde{X}_n$ often not psd!

* Solve SDP as decision problem

  $\Rightarrow X_1, \dots, X_n$ (approx.) in the relative interior of opt. face

* Note: interior of psd cone is cone of pos. definite matrices

* Remaining near zero eigenvalues are probably there for a (structural) reason

$\Rightarrow$ We expect there to be rational bases of the kernel with small denominators

$\boxed{\text{Add these constraints before rounding } \Rightarrow \tilde{X}_1, \dots, \tilde{X}_n \succeq 0}$

$\boxed{\text{Rational bases for the kernel}}$

* Use high precision floating point arithmetic to compute N_i containing the kernel vectors of X_i as columns. Let $c_i = \#\mathrm{Columns}(N_i)$, $r_i = \#\mathrm{Rows}(N_i)$

Rational bases for the kernel

* Use high precision floating point arithmetic to compute $\mathcal{N}_i$ containing the kernel vectors of X_i as columns. Let $c_i = \#\mathrm{Columns}(\mathcal{N}_i)$, $r_i = \#\mathrm{Rows}(\mathcal{N}_i)$

* While $r_i - c_i > \mathrm{Rows}(A)$:
  - Use LLL algorithm to find integer relations between rows of $\mathcal{N}_i$ (we use Nemo)

Rational bases for the kernel

* Use high precision floating point arithmetic to compute $\mathcal{N}_i$ containing the kernel vectors of X_i as columns. Let $c_i = \#\text{Columns}(\mathcal{N}_i)$, $r_i = \#\text{Rows}(\mathcal{N}_i)$

* While $r_i - c_i > \text{Rows}(A)$:
  - Use LLL algorithm to find integer relations between rows of $\mathcal{N}_i$ (we use Nemo)
  - Remove row from $\mathcal{N}_i$ whose coefficient in the relation is nonzero

Rational bases for the kernel

* Use high precision floating point arithmetic to compute $\mathcal{N}_i$ containing the kernel vectors of X_i as columns. Let $c_i = \#\mathrm{Columns}(\mathcal{N}_i)$, $r_i = \#\mathrm{Rows}(\mathcal{N}_i)$

* While $r_i - c_i > \mathrm{Rows}(A)$:
    - Use LLL algorithm to find integer relations between rows of $\mathcal{N}_i$ (we use Nemo)
    - Remove row from $\mathcal{N}_i$ whose coefficient in the relation is nonzero
    - Matrix A in reduced row echelon form whose rows are the relations we found

Rational bases for the kernel

* Use high precision floating point arithmetic to compute $\mathcal{N}_i$ containing the kernel vectors of X_i as columns. Let $c_i = \#\text{Columns}(\mathcal{N}_i)$, $r_i = \#\text{Rows}(\mathcal{N}_i)$

* While $r_i - c_i > \text{Rows}(A)$:
  - Use LLL algorithm to find integer relations between rows of $\mathcal{N}_i$ (we use Nemo)
  - Remove row from $\mathcal{N}_i$ whose coefficient in the relation is nonzero
  - Matrix A in reduced row echelon form whose rows are the relations we found
  - If no new relation found: STOP

$\boxed{\text{Rational bases for the kernel}}$

* Use high precision floating point arithmetic to compute N_i containing the kernel vectors of X_i as columns. Let $c_i = \#\text{Columns}(N_i)$, $r_i = \#\text{Rows}(N_i)$

* While $r_i - c_i > \text{Rows}(A)$:
  - Use LLL algorithm to find integer relations between rows of N_i (we use Nemo)
  - Remove row from N_i whose coefficient in the relation is nonzero
  - Matrix A in reduced row echelon form whose rows are the relations we found
  - If no new relation found: STOP

* Nullspace of A (Computable in exact arithmetic) contains a basis for the kernel of X_i

Rational bases for the kernel

* Use high precision floating point arithmetic to compute N_i containing the kernel vectors of X_i as columns. Let $c_i = \#\mathrm{Columns}(N_i)$, $r_i = \#\mathrm{Rows}(N_i)$

* While $r_i - c_i > \mathrm{Rows}(A)$:
  - Use LLL algorithm to find integer relations between rows of N_i (we use Nemo)
  - Remove row from N_i whose coefficient in the relation is nonzero
  - Matrix A in reduced row echelon form whose rows are the relations we found
  - If no new relation found: STOP

* Nullspace of A (Computable in exact arithmetic) contains a basis for the kernel of X_i

For each kernel vector v add $X_i v = 0$ to lin. system

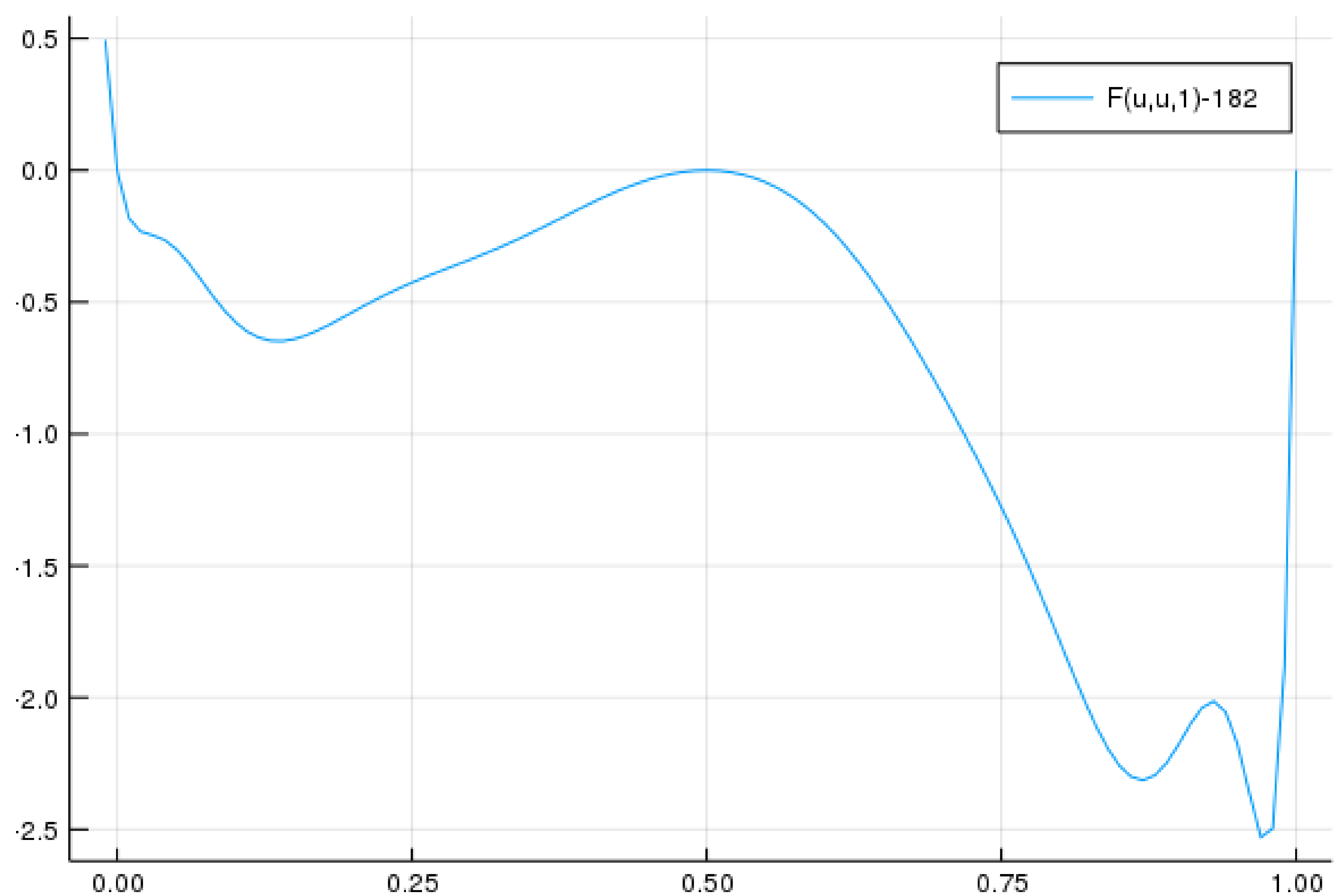
Uniqueness

Claim: For any feasible configuration C with $|C| = 183$: $\{c \cdot c' : c, c' \in C, c \neq c'\} = \{-1, -1/2, 0, 1/2\}$.

Uniqueness

Claim: For any feasible configuration C with $|C| = 183$: $\{c \cdot c' : c, c' \in C, c \neq c'\} = \{-1, -\tfrac{1}{2}, 0, \tfrac{1}{2}\}$.

determined rational solution $F(u,v,t)$



Complementary Slackness

$F(e \cdot c, e \cdot c, 1) = 182$ for $c \in C$

$\Rightarrow \forall C: \; e \cdot c \in \{0, \tfrac{1}{2}, 1\}$ for all $c \in C$

Uniqueness

$F(u,u,1) = 182$ for $u \in \Delta_\circ$ if and only if $u \in \{0, 1/2, 1\}$.

$\Rightarrow$ for all configurations C with $|C| = 183$: $\{e \cdot c : c \in C\} = \{0, 1/2, 1\}$

Complementary Slackness $F(e\cdot c, e\cdot c', c\cdot c') = -1$ for $c, c' \in C$, $c \neq c'$

$F(e\cdot c, e\cdot c, 1) = 182$ for $c \in C$

Uniqueness

$F(u,u,1) = 182$ for $u \in \Delta_0$ if and only if $u \in \{0, 1/2, 1\}$.

$\Rightarrow$ for all configurations C with $|C| = 183$: $\{e \cdot c : c \in C\} = \{0, 1/2, 1\}$

Complementary Slackness $F(e \cdot c, e \cdot c', c \cdot c') = -1$ for $c, c' \in C$, $c \neq c'$

$F(e \cdot c, e \cdot c, 1) = 182$ for $c \in C$

For all $u_0, v_0 \in \{0, 1/2, 1\}$ check zeros of $F(u_0, v_0, t) + 1$ for $(u_0, v_0, t) \in \Delta$.

For any feasible configuration C with $|C| = 183$: $\{c \cdot c' : c, c' \in C, c \neq c'\} = \{-1, -1/2, 0, 1/2\}$.

$\boxed{\text{Uniqueness}}$

Let C be an optimal configuration $\Rightarrow$ $c \cdot c' \in \{0, \pm 1/2, \pm 1\}$ for all $c, c' \in C$

$\|c\| = 1$ for all $c \in C$

Uniqueness

Let C be an optimal configuration $\Rightarrow$ $c \cdot c' \in \{0, \pm 1/2, \pm 1\}$ for all $c, c' \in C$

$\|c\| = 1$ for all $c \in C$

Rescale all $c \in C$ s.t. $\|c\| = \sqrt{2}$

L = additive group generated by C $\Rightarrow$ L is a root lattice

$\Rightarrow$ L sum of A_n, D_n, or E_n

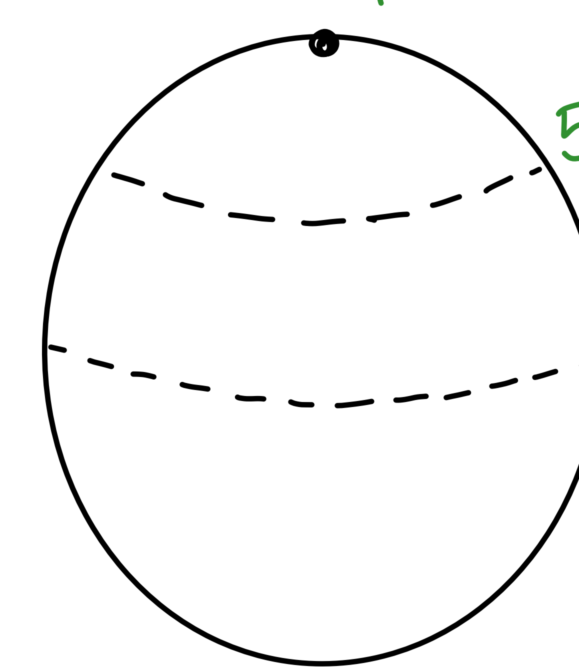
Only sum with at least 183 minimal vectors is E_8.

Uniqueness

Let C be an optimal configuration $\Rightarrow c\cdot c' \in \{0, \pm 1/2, \pm 1\}$ for all $c, c' \in C$
$\|c\| = 1$ for all $c \in C$

Rescale all $c \in C$ s.t. $\|c\| = \sqrt{2}$

L = additive group generated by C $\Rightarrow L$ is a root lattice
$\Rightarrow L$ sum of A_n, D_n, or E_n



Only sum with at least 183 minimal vectors is E_8.

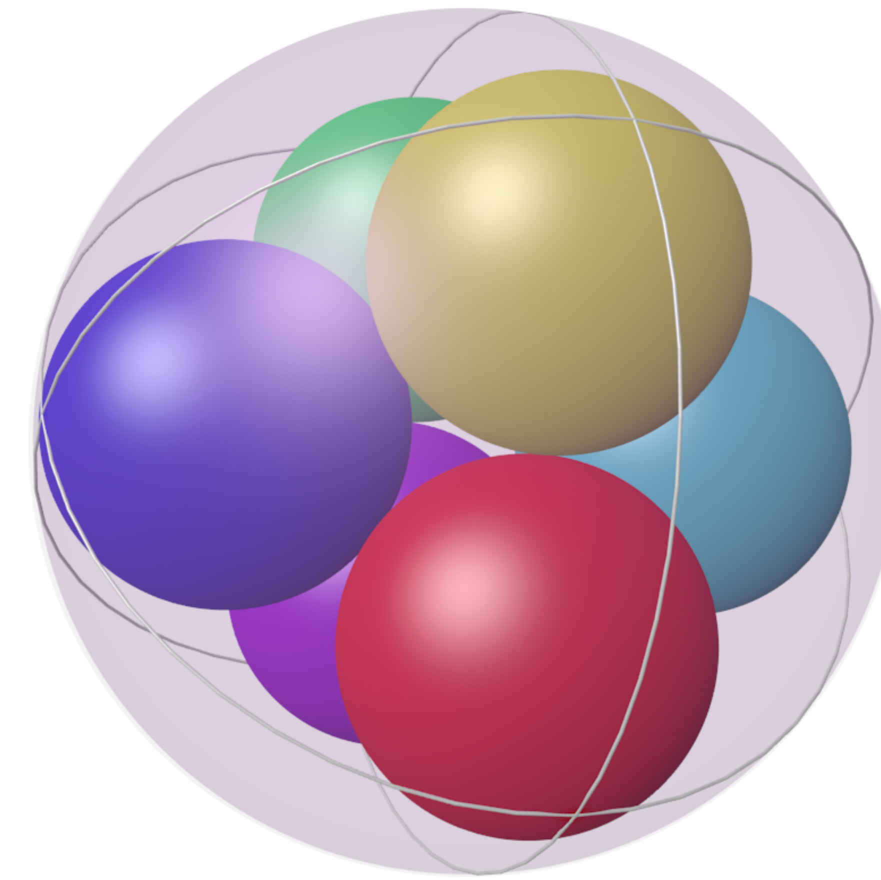
$183 - 57 = 126$ points on equator

Since for all $c \in C$: $e\cdot c \in \{0, 1/2, 1\}$ & there exists no $(7, 57, 1/3)$ spherical code

$\Rightarrow C$ has to be the configuration we know!

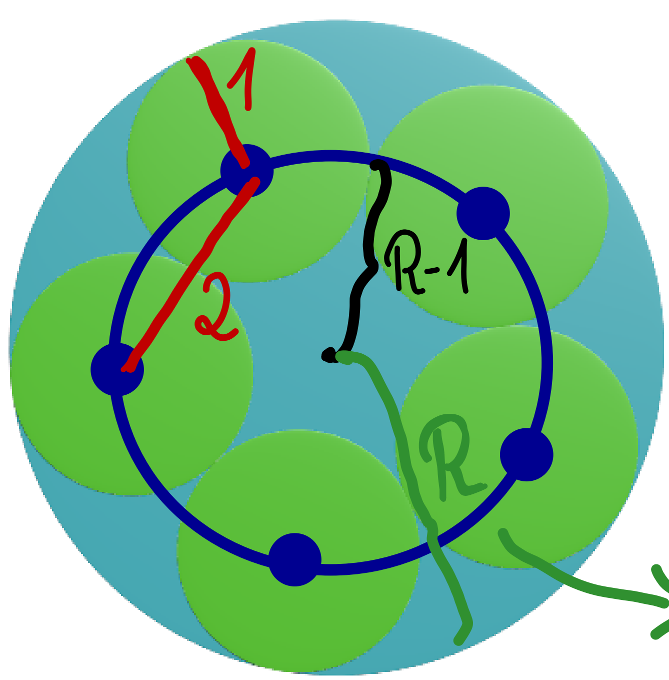
Packing unit spheres in a larger sphere

Packing unit spheres in sphere of radius $R \geq 1$



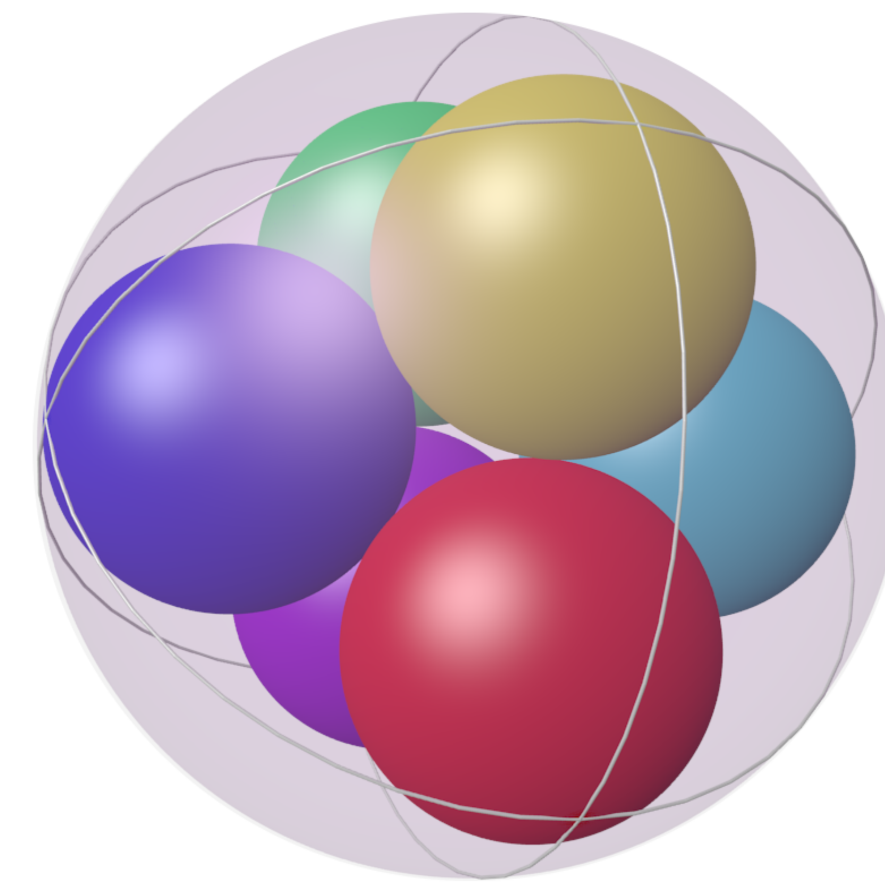
Packing unit spheres in a larger sphere

Packing unit spheres in sphere of radius $R \geq 1$



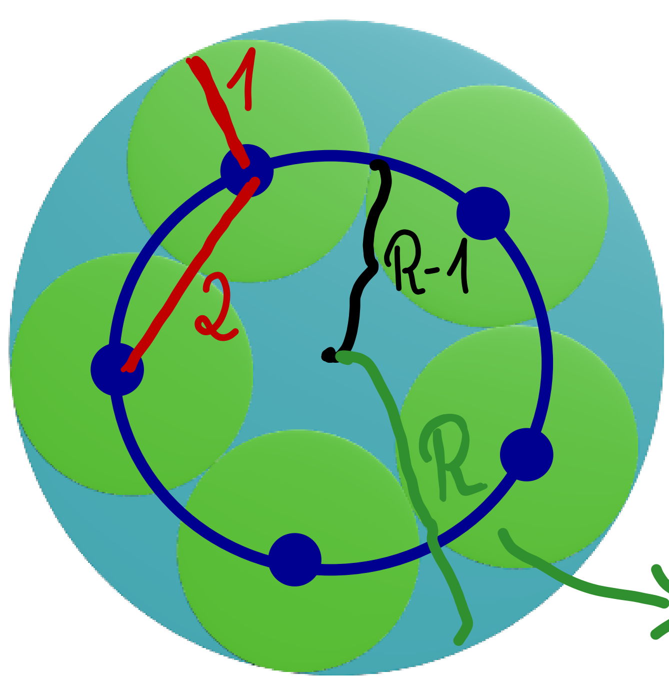
$\Rightarrow$ codes in balls of radius $R-1$ where minimal distance between points is 2.

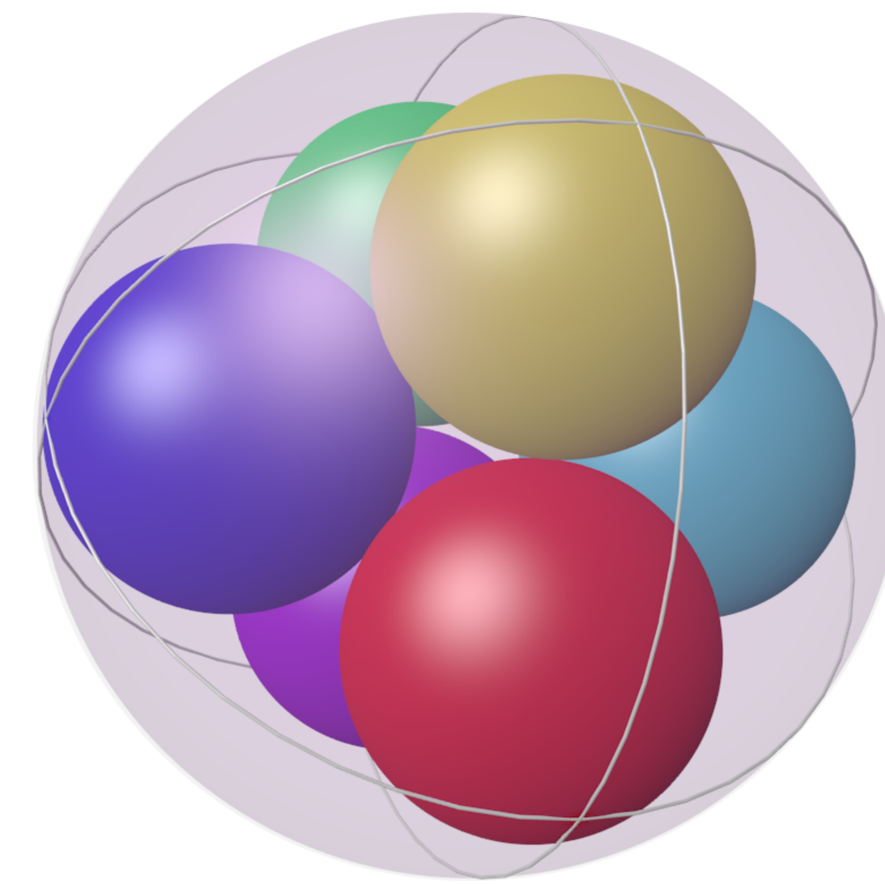
$R = 1 + \sqrt{2\left(1 + \frac{1}{\sqrt{5}}\right)}$



Packing unit spheres in a larger sphere

Packing unit spheres in sphere of radius $R \geq 1$





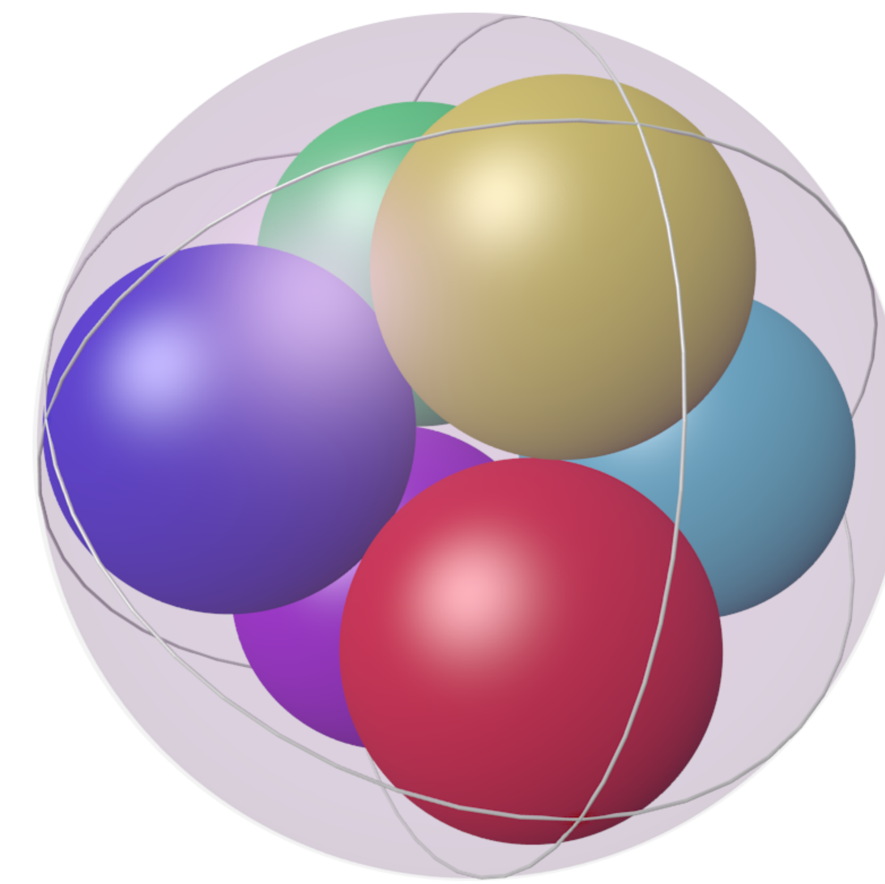
$\Rightarrow$ codes in balls of radius $R-1$ where minimal distance between points is 2.

$R = 1 + \sqrt{2(1 + 1/\sqrt{5})}$

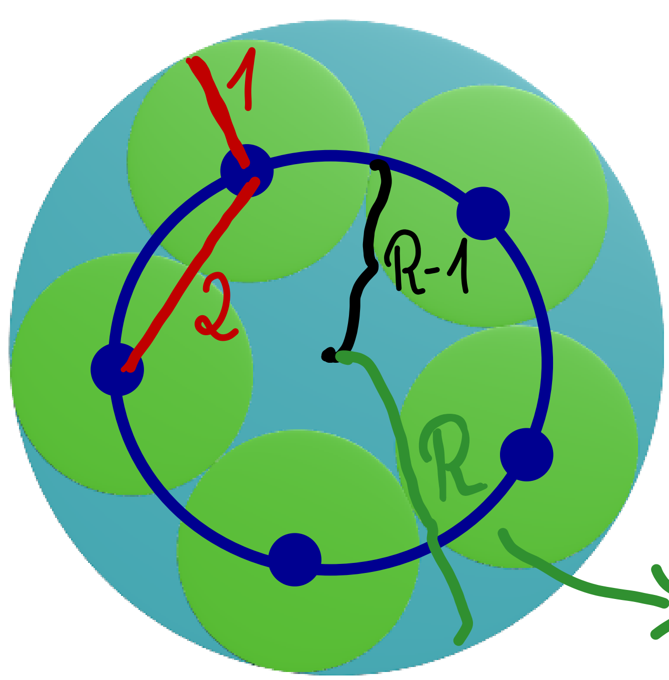
$\Rightarrow$ Upper bound via SDP which is similar to previous SDP

Packing unit spheres in a larger sphere

Packing unit spheres in sphere of radius $R \geq 1$



$\Rightarrow$ codes in balls of radius $R-1$ where minimal distance between points is 2.



$\Rightarrow R = 1 + \sqrt{2(1 + 1/\sqrt{5})}$

$\Rightarrow$ Upper bound via SDP which is similar to previous SDP

Problem: $R-1$ often irrational

$\boxed{\text{Quadratic Field}}$

- Approximate solution $x^* = \bigoplus_{i=1}^{m} \mathrm{vec}(X_i^*)$ for $Ax^* = b, X_i^* \succcurlyeq 0$
- Assume there exists a solution vector over $\mathbb{Q}[\sqrt{s}]$

Quadratic Field

- Approximate solution $x^* = \bigoplus_{i=1}^{m} \mathrm{vec}(X_i^*)$ for $Ax^* = b, X_i^* \succcurlyeq 0$
- Assume there exists a solution vector over $\mathbb{Q}[\sqrt{5}]$
- Compute x_1^*, x_2^* in floating point:

$$(A_1 + \sqrt{5}A_2)(x_1^* + \sqrt{5}x_2^*) = b_1 + \sqrt{5}b_2$$

Quadratic Field

- Approximate solution $x^* = \bigoplus_{i=1}^{m} \mathrm{vec}(X_i^*)$ for $Ax^* = b, X_i^* \succeq 0$
- Assume there exists a solution vector over $\mathbb{Q}[\sqrt{s}]$
- Compute x_1^*, x_2^* in floating point:

$$(A_1 + \sqrt{s}A_2)(x_1^* + \sqrt{s}x_2^*) = b_1 + \sqrt{s}b_2$$

$$\Leftrightarrow \quad \begin{pmatrix} A_1 & sA_2 \\ A_2 & A_1 \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$\boxed{\text{Quadratic Field}}$

- Approximate solution $x^* = \bigoplus_{i=1}^{m} \mathrm{vec}(X_i^*)$ for $Ax^* = b, X_i^* \succeq 0$
- Assume there exists a solution vector over $\mathbb{Q}[\sqrt{s}]$
- Compute x_1^*, x_2^* in floating point:

$$(A_1 + \sqrt{s}A_2)(x_1^* + \sqrt{s}x_2^*) = b_1 + \sqrt{s}b_2$$

$$\Leftrightarrow \quad \begin{pmatrix} A_1 & sA_2 \\ A_2 & A_1 \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

- Now problem over $\mathbb{Q}$

$\Rightarrow$ rounding and kernel detection over $\mathbb{Q}$

Exact SDP Bounds

- General rational solution for packing
  - 3 spheres in a sphere with $R = 1 + \frac{2}{\sqrt{2}}$ for all dimensions
  - $2d$ spheres in a sphere with $R = 1 + \sqrt{2}$ for all dimensions
  - $d+1$ spheres in a sphere with $R = 1 + \sqrt{2d/(d+1)}$ for all dimensions
- For $d=2$: rational solution for packing 7 spheres in larger spheres
- Reprove uniqueness of 10 points in S^3 (rational solution)
- Prove uniqueness kissing configuration on hemisphere in dim 8 (rational solution)
- For $d=2$: solution in $\mathbb{Q}[\sqrt{5}]$ for packing 5 spheres in larger spheres

Next goal: Prove uniqueness of 8 points in S^2 (solution in $\mathbb{Q}[\sqrt{8}]$)

Thank you!

Thank you!

Summer School in
Optimization, Interpolation & Modular Forms
24-28 Aug 2020 at EPFL, Lausanne
(Organizers: Maryna Viazovska, Matthew de Courcy-Ireland, M.D.)