Efficient linear computation of the characteristic polynomials of the p-curvatures of a differential operator with integer coefficients.

Seminar LFANT

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November 23, 2020

Motivating the p-curvature

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If this system admits an algebraic basis of solutions then its reduction modulo p has an algebraic basis of solutions for almost all p prime.

Grothendieck-Katz conjecture : This implication is in fact an equivalence.

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- *The dimension of the kernel of the p-curvature of this system.*

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When the connexion on M is of the form ∂_A then

$$A_0 = I_n \qquad A_{k+1} = A_k' - AA_k \qquad A_p$$

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$$\chi(A_p) = x^3 + \frac{2}{z^3+1}x + \frac{z^6+2z^3}{z^3+1}$$

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$$\partial x = x\partial + 1$$

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- Contribution : $L \in \mathbb{Z}(x)\langle\partial\rangle$. Computation of all the characteristic polynomials of its p-curvatures for $p \leqslant N$ in $\tilde{O}(N)$ binary operations.

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- Computation of $(p-1)! \mod p^s$ for all $p \leqslant N$: $\tilde{O}(sN)$ binary operations [Costa, Gerbicz, Harvey, 2014].

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Example

$(x+1)\partial$ invertible in $k(x)\langle\partial^{\pm 1}\rangle$
$\partial + \theta$ non invertible in $k(\theta)\langle\partial^{\pm 1}\rangle$

$\Xi_{x,\partial}$ and $\Xi_{\theta,\partial}$

Let $L' = l_m{}'(\theta)\partial^m + \ldots + l_1{}'(\theta)\partial + l_0{}'(\theta)$.

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$$B(L') = \begin{pmatrix} & & & -\frac{l_0{}'}{l_m{}'} \\ 1 & & & -\frac{l_1{}'}{l_m{}'} \\ & \ddots & & \vdots \\ & & 1 & -\frac{l_{m-1}{}'}{l_m{}'} \end{pmatrix}$$

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Let $L = l_m(x)\partial^m + \ldots + l_1(x)\partial + l_0(x)$.

$$\Xi_{x,\partial}(L) = l_m(x)^p \chi(A_p(L))(\partial^p)$$

$$\Xi_{\theta,\partial}(L') = \left(\prod_{i=0}^{p-1} l_m{}'(\theta+i)\right) \chi(B_p(L'))(\partial^p)$$

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Lemma

- *Send an irreducible element over a power of an irreductible element of the center*

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Lemma

- *Send an irreducible element over a power of an irreductible element of the center*
- *Multiplicative.*

$\Xi_{x,\partial}$ and $\Xi_{\theta,\partial}$

Theorem

The followin diagram commutes.

$$\begin{array}{ccc} k[x]\langle\partial^{\pm 1}\rangle & \xrightarrow{\sim} & k[\theta]\langle\partial^{\pm 1}\rangle \\ \downarrow \Xi_{x,\partial} & & \downarrow \Xi_{\theta,\partial} \\ k[x^p][\partial^{\pm p}] & \xrightarrow{\sim} & k[\theta^p - \theta][\partial^{\pm p}] \end{array}$$

$\Xi_{x,\partial}$ and $\Xi_{\theta,\partial}$

$$(x^2 + 2x + 1)\partial^3 - x\partial + x^3 + 3$$

$\Xi_{x,\partial}$ and $\Xi_{\theta,\partial}$

$$(x^2+2x+1)\partial^3 - x\partial + x^3 + 3 \quad \mapsto \quad \begin{pmatrix} \partial^6+2\theta\partial^5+(\theta^2-\theta)\partial^4 \\ -(\theta+3)\partial^3+(\theta^3-3\theta^2+2\theta) \end{pmatrix} \partial^{-3}$$

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$$\begin{pmatrix} & & -\frac{x^3+3}{x^2+2x+1} \\ 1 & & \frac{x}{x^2+2x+1} \\ & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} & & & & & -(\theta^3-3\theta^2+2\theta) \\ 1 & & & & & 0 \\ & 1 & & & & 0 \\ & & 1 & & & (\theta+3) \\ & & & 1 & & -(\theta^2-\theta) \\ & & & & 1 & -2\theta \end{pmatrix}$$

proof of the commutativity

- Step 1 : Isomorphism with a matrix algebra after scalar extension.

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$$\mathcal{M}_\theta(\theta) = \begin{pmatrix} T & & & \\ & T+1 & & \\ & & \ddots & \\ & & & T+p-1 \end{pmatrix} \text{ and } \mathcal{M}_\theta(\partial) = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \partial^p & & & \end{pmatrix}$$

Step 2 : The determinant, restriction, corestriction

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\mathcal{D}_x[T] & \xrightarrow{\mathcal{M}_x} & M_p(\mathcal{Z}_x[T]) & \xrightarrow{\det} & \mathcal{Z}_x[T] \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
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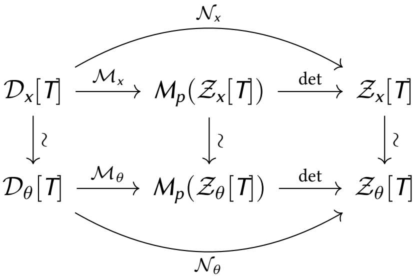

$\mathcal{N}_\cdot(\mathcal{D}_\cdot) \subset \mathcal{Z}_\cdot$ Invariance by $T \mapsto T + a$

Step 3 : Equality with the determinant

Lemma

- $\mathcal{N}_{\cdot}(L)$ *and* $\Xi_{\cdot,\partial}(L)$ *have the same leading coefficient.*

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Size of the output at the end of step 2 : $O(N^2)$

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$\forall i \leqslant p - 1$

$$p_i = (-1)^i p'_i$$

Structure of the algorithm

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Computation of $(p-1)! \mod p^s$ [Costa, Gerbicz, Harvey, 2014]

$N = 7.$

$$(3-1)!$$

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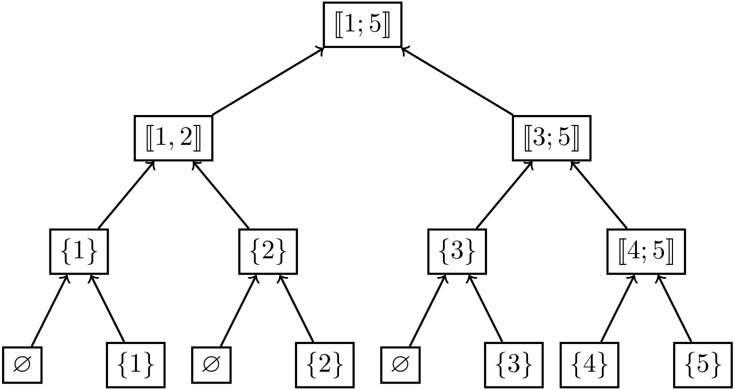

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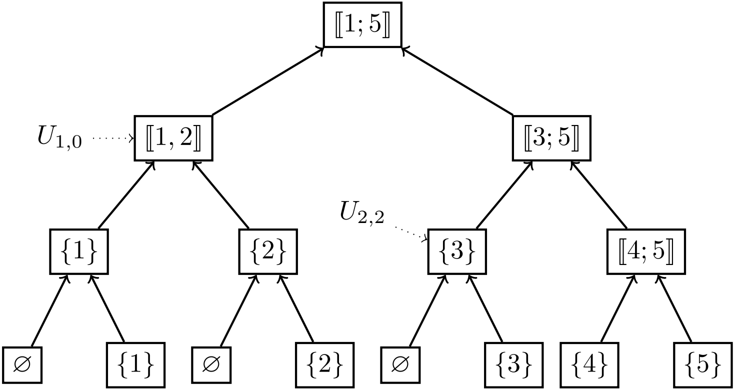

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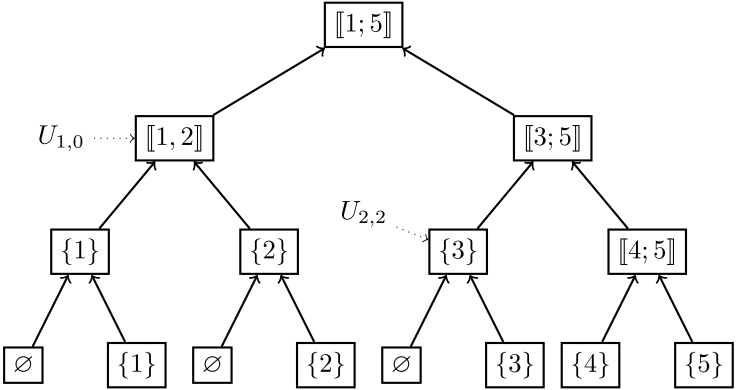

$U_{i,j} = U_{i+1,2j} \amalg U_{i+1,2j+1} \quad A_{i,j} = \prod_{k \in U_{i,j}} k \quad S_{i,j} = \prod_{\substack{p \in U_{i,j} \\ p \text{ prime}}} p^s$

Computation of $(p-1)! \mod p^s$ [Costa, Gerbicz, Harvey, 2014]

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$\tilde{O}(sN)$ binary operations.

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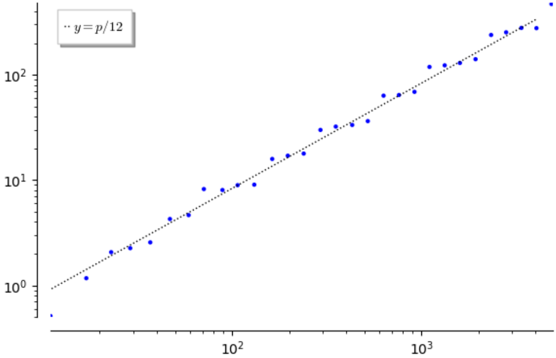
- Deduce the $\chi(A_p(L))$.
  $\tilde{O}(N)$

Implementation

Implementation of the algorithm



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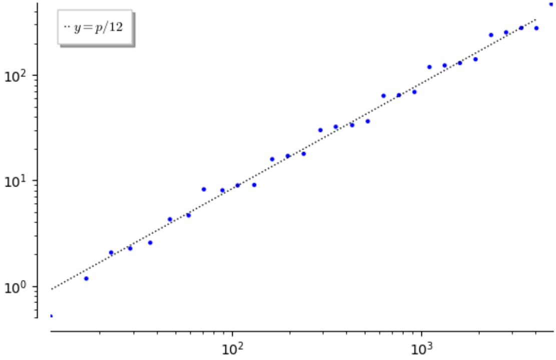

Complexity : If $L \in \mathbb{Z}[x]\langle\partial\rangle$ is of degree m, has polynomial coefficients of degree at most d and has integer coefficients of size at most n then :

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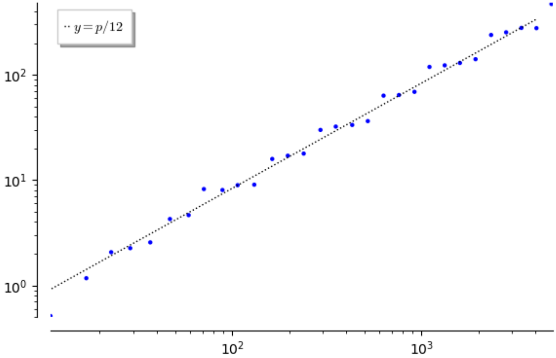

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$$O(Nd((n+d)(m+d)^{w}+(m+d)^{\Omega})).$$

Future works

- [Bostan, Caruso, Schost, 2016] brought the computation of invariant factors of the p-curvature to that of a factorial.

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- [Bostan, Caruso, Schost, 2016] brought the computation of invariant factors of the p-curvature to that of a factorial.
- Extension to operators with coefficients in a number field.