On the hardness of code equivalence problem in rank metric

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1. Introduction

2. In Hamming metric

3. In rank metric
   - The code equivalence problem for $\mathbb{F}_{q^m}$–linear codes
   - The general case

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The problem

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The metric may be:

- Euclidean metric (lattice isometry/similitude problem);
- Hamming metric (code permutation/monomial equivalence problem);
- **Today's purpose** rank metric.

Code equivalence problems in Hamming metric

Problem 2 (Permutation Equivalence of Codes (PEC))

Let $\mathscr{C}_1, \mathscr{C}_2 \subseteq \mathbb{F}_q^n$ *be two codes. Decide whether there exists* $\boldsymbol{P} \in \mathfrak{S}_n$ *such that*

$$\mathscr{C}_1 = \mathscr{C}_2 \cdot \boldsymbol{P}.$$

Problem 3 (Monomial Equivalence of Codes (MEC))

Let $\mathscr{C}_1, \mathscr{C}_2 \subseteq \mathbb{F}_q^n$ *be two codes. Decide whether there exists* $\boldsymbol{P} \in \mathfrak{S}_n$ *and* $\boldsymbol{D} \in Diag(n)$ *such that*

$$\mathscr{C}_1 = \mathscr{C}_2 \cdot \boldsymbol{D} \cdot \boldsymbol{P}.$$

Previous works on equivalence in Hamming metric

- **Theoretical results**
  - Code equivalence is harder than graph isomorphism (Petrank, Roth, 1997);
  - Code equivalence is **not** $\mathcal{NP}$–hard (unless polynomial time hierarchy collapses)
  - If $\mathscr{C} \cap \mathscr{C}^{\perp} = \{0\}$ then code equivalence is as hard as graph isomorphism (Bardet, Otmani, Saeed 2019)

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- **Algorithms**
  - Leon (1982);
  - Sendrier (2000) solves equivalence in $O(2^{\dim \mathscr{C} \cap \mathscr{C}^\perp} n^\omega)$;
  - Feulner (2009), techniques from symmetric cryptography;
  - Saeed (2017), Gröbner bases.

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Matrix codes

The space of $m \times n$ matrices with entries in $\mathbb{F}_q$ is denoted by $\mathcal{M}_{m,n}(\mathbb{F}_q)$.

Definition 1

A matrix code is a subspace $\mathscr{C}^{mat}$ of $\mathcal{M}_{m,n}(\mathbb{F}_q)$ endowed with the rank metric *:*

$$d_R(\boldsymbol{A}, \boldsymbol{B}) = Rk\,(\boldsymbol{A} - \boldsymbol{B}).$$

Vector codes

- Fix an $\mathbb{F}_q$–basis $\mathcal{B}$ of $\mathbb{F}_{q^m}$. Then, to any subspace $\mathscr{C} \subseteq \mathbb{F}_{q^m}^n$ corresponds a matrix code

$$\mathscr{C}^{\mathrm{mat}} \subseteq \mathcal{M}_{m,n}(\mathbb{F}_q).$$

- Conversely, let a be a primitive element of $\mathbb{F}_{q^m}/\mathbb{F}_q$ and $C(a)$ the matrix representing the $\mathbb{F}_q$–linear map $x \mapsto ax$ in a basis $\mathcal{B}$. A matrix code $\mathscr{C}^{\mathrm{mat}}$ such that

$$C(a) \cdot \mathscr{C}^{\mathrm{mat}} \subseteq \mathscr{C}^{\mathrm{mat}}$$

comes from a vector code.

Stabilizer algebras

Definition 2

Let $\mathscr{C} \subseteq \mathcal{M}_{m,n}(\mathbb{F}_q)$ be a matrix code. The left *(resp.* right*)* stabilizer algebra *of $\mathscr{C}$ is defined as*

$$\mathrm{Stab}_L(\mathscr{C}) \overset{def}{=} \{\boldsymbol{P} \in \mathcal{M}_m(\mathbb{F}_q) \mid \boldsymbol{P} \cdot \mathscr{C} \subseteq \mathscr{C}\}$$

$$\text{resp.} \quad \mathrm{Stab}_R(\mathscr{C}) \overset{def}{=} \{\boldsymbol{Q} \in \mathcal{M}_n(\mathbb{F}_q) \mid \mathscr{C} \cdot \boldsymbol{Q} \subseteq \mathscr{C}\}$$

Lemma 1

A matrix code $\mathscr{C} \subseteq \mathcal{M}_{m,n}(\mathbb{F}_q)$ whose left stabilizer algebra contains a representation of $\mathbb{F}_{q^m}$ is $\mathbb{F}_{q^m}$–linear.

Rank–preserving linear maps

Theorem 1

The group of linear automorphisms $\phi : \mathcal{M}_{m,n}(\mathbb{F}_q) \to \mathcal{M}_{m,n(\mathbb{F}_q)}$ *preserving the ranks is spanned by the maps:*

- $\boldsymbol{X} \mapsto \boldsymbol{A} \cdot \boldsymbol{X}$ *for some* $\boldsymbol{A} \in \mathrm{GL}_m(\mathbb{F}_q)$*;*
- $\boldsymbol{X} \mapsto \boldsymbol{X} \cdot \boldsymbol{B}$ *for some* $\boldsymbol{B} \in \mathrm{GL}_n(\mathbb{F}_q)$*;*
- *(only for* $m = n$*):* $\boldsymbol{X} \mapsto \boldsymbol{X}^T$*.*

Equivalence problem in rank metric

Problem 4 (Rank Equivalence of Matrix Codes (REMC))

Given $\mathscr{C}_1^{mat}, \mathscr{C}_2^{mat} \in \mathcal{M}_{m,n}(\mathbb{F}_q)$, *decide wheter there exists* $\boldsymbol{P} \in \mathsf{GL}_m(\mathbb{F}_q)$ *and* $\boldsymbol{Q} \in \mathsf{GL}_n(\mathbb{F}_q)$ *such that*

$$\mathscr{C}_1^{mat} = \boldsymbol{P} \cdot \mathscr{C}_2^{mat} \cdot \boldsymbol{Q}.$$

Our contribution

Theorem 2 (C., Debris–Alazard, Gaborit, 2020)

For $\mathbb{F}_{q^m}$–linear codes $\mathscr{C}_1^{vec}, \mathscr{C}_2^{vec} \subseteq \mathbb{F}_{q^m}^n$, the equivalence problem in rank metric is in $\mathcal{P}$ if $q = (mn)^{O(1)}$. Else it is in $\mathcal{ZPP}$.

Theorem 3 (C., Debris–Alazard, Gaborit, 2020)

For general matrix spaces, the equivalence problem in rank metric is harder than the equivalence problem in Hamming metric.

Statement

If the vector structure is known:

Problem 5 (Rank Equivalence of Vector Codes (REVC))

Given $\mathscr{C}_1, \mathscr{C}_2 \subseteq \mathbb{F}_{q^m}^n$, decide whether there exists $\boldsymbol{P} \in \mathrm{GL}_n(\mathbb{F}_q)$ such that

$$\mathscr{C}_1 = \mathscr{C}_2 \cdot \boldsymbol{P}$$

If not:

Problem 6 (Rank Equivalence of Hidden Vector Codes (REHVC))

Given $\mathscr{C}_1^{mat}, \mathscr{C}_2^{mat} \subseteq \mathcal{M}_{m,n}(\mathbb{F}_q)$ constructed from $\mathbb{F}_{q^m}$–linear codes with possibly distinct bases. Decide whether there exists $\boldsymbol{P} \in \mathrm{GL}_m(\mathbb{F}_q)$ and $\boldsymbol{Q} \in \mathrm{GL}_n(\mathbb{F}_q)$ such that

$$\mathscr{C}_1^{mat} = \boldsymbol{P} \cdot \mathscr{C}_2^{mat} \cdot \boldsymbol{Q}.$$

Reducing to another problem

Even when the vector structure is hidden, $\boldsymbol{P}$ may be recovered separately by computing the left stabilizer algebras which are conjugated under $\boldsymbol{P}$. Hence we are reduced to:

Problem 7 (Right equivalence)

Given matrix codes $\mathscr{C}_1^{mat}, \mathscr{C}_2^{mat} \subseteq \mathcal{M}_{m,n}(\mathbb{F}_q)$, *decide whether there exists* $\boldsymbol{P} \in \mathrm{GL}_n(\mathbb{F}_q)$ *such that*

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Fact

Finding the space of $\boldsymbol{P} \in \mathcal{M}_n(\mathbb{F}_q)$ *such that* $\mathscr{C}_2^{mat} \cdot \boldsymbol{P} \subseteq \mathscr{C}_1^{mat}$ *boils down to solve a linear system.*

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But, what if the space of solutions contains singular matrices? How to decide whether there is a nonsingular one in it?

Considering the right stabilizer algebras

Theorem 4

Let $\mathscr{C}_1^{mat}, \mathscr{C}_2^{mat} \in \mathcal{M}_{m,n}(\mathbb{F}_q)$ *such that* $\mathrm{Stab}_R(\mathscr{C}_1^{mat})$ *is a division algebra. If there exists* $\boldsymbol{Q} \in \mathrm{GL}_n(\mathbb{F}_q)$ *such that*

$$\mathscr{C}_1^{mat} = \mathscr{C}_2^{mat} \cdot \boldsymbol{Q}$$

then any $\boldsymbol{P} \in \mathcal{M}_n(\mathbb{F}_q)$ *such that* $\mathscr{C}_1^{mat} \cdot \boldsymbol{P} \subseteq \mathscr{C}_2^{mat}$ *is nonsingular.*

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Proof.

Suppose that $\exists \boldsymbol{P}$ singular such that $\mathscr{C}_1^{\mathrm{mat}} \cdot \boldsymbol{P} \subseteq \mathscr{C}_2^{\mathrm{mat}}$. Then

$$\mathscr{C}_1^{\mathrm{mat}} \cdot \boldsymbol{P} \cdot \boldsymbol{Q} \subseteq \mathscr{C}_2^{\mathrm{mat}} \cdot \boldsymbol{Q} = \mathscr{C}_1^{\mathrm{mat}}$$

Hence $\boldsymbol{PQ} \in \mathrm{Stab}_R(\mathscr{C}_1^{\mathrm{mat}})$ and is singular: a contradiction. $\square$

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Consequence. In this situation, the problem is easy to solve:

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1. Compute the space of $\boldsymbol{P} \in \mathcal{M}_n(\mathbb{F}_q)$ such that $\mathscr{C}_1^{\mathrm{mat}} \cdot \boldsymbol{P} \subseteq \mathscr{C}_2^{\mathrm{mat}}$;

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Consequence. In this situation, the problem is easy to solve:

1. Compute the space of $\boldsymbol{P} \in \mathcal{M}_n(\mathbb{F}_q)$ such that $\mathscr{C}_1^{\mathrm{mat}} \cdot \boldsymbol{P} \subseteq \mathscr{C}_2^{\mathrm{mat}}$;
2. Pick a nonzero element $\boldsymbol{P}$ in the solution space:
   - if $\boldsymbol{P}$ is singular, then the codes are **not** right equivalent;
   - else they are right equivalent and $\boldsymbol{P}$ realizes the equivalence.

About finite dimensional algebras

A subalgebra $\mathcal{A} \subseteq \mathcal{M}_n(\mathbb{F}_q)$ is

- **simple** if it has no nontrivial two–sided ideals. Artin Wedderburn theory $\Rightarrow$ any simple algebra over $\mathbb{F}_q$ are isomorphic to $\mathcal{M}_r(\mathbb{F}_{q^\ell})$ for some r, ℓ.
- **semi-simple** if it is isomorphic to a cartesian product of simple algebras.

Definition 3 (Jacobson radical)

The radical of an algebra $\mathcal{A}$ is defined as

$$\mathrm{Rad}(\mathcal{A}) \stackrel{def}{=} \{\boldsymbol{N} \in \mathcal{A} \mid \forall \boldsymbol{M} \in \mathcal{A},\ \boldsymbol{MN} \text{ is nilpotent}\}$$

Theorem 5

$\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ *is semi–simple.*

A picture

About finite dimensional algebras – algorithms

- Friedl, Rónyai 1985: the Jacobson radical and the Artin Wedderburn decomposition can be computed in polynomial time. Their algorithm rests on two tools:
  - linear algebra;
  - factorisation of univariate polynomials (this is the why of $\mathcal{P}$ v.s. $\mathcal{ZPP}$).
- Rónyai 1990. Given a simple algebra the isomorphism with $\mathcal{M}_r(\mathbb{F}_{q^\ell})$ can be explicitly computed.

Framework for solving right equivalence

Input. Two matrix codes $\mathscr{C}_1^{\mathrm{mat}}, \mathscr{C}_2^{\mathrm{mat}} \subseteq \mathcal{M}_{m,n}(\mathbb{F}_q)$.

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- Compute the Artin–Wedderburn decomposition of $\mathrm{Stab}_R(\mathscr{C}_1^{\mathrm{mat}})/\mathrm{Rad}(\mathrm{Stab}_R(\mathscr{C}_1^{\mathrm{mat}}))$, deduce a decomposition of

$$1 = e_1 + \cdots + e_r$$

as a sum of minimal orthogonal idempotents; lift idempotents (effective Wedderburn Malcev) and compare the codes

$$\mathscr{C}_1^{\mathrm{mat}} e_1, \ldots, \mathscr{C}_1^{\mathrm{mat}} e_r$$

with the corresponding codes from $\mathscr{C}_2^{\mathrm{mat}}$.

A picture

A picture

The general problem

Theorem 6

The general rank equivalence of matrix codes (REMC) problem is harder than the Hamming metric monomial equivalence problem.

Sketch of proof of the reduction

Let $\mathscr{C}_1, \mathscr{C}_2 \subseteq \mathbb{F}_{q^m}$ with generator matrices $\boldsymbol{G}_1, \boldsymbol{G}_2$.

$$\boldsymbol{G}_1 = \left(\begin{array}{c|c|c|c} \boldsymbol{c}_1^\top & \boldsymbol{c}_2^\top & \cdots & \boldsymbol{c}_n^\top \end{array} \right), \quad \boldsymbol{G}_2 = \left(\begin{array}{c|c|c|c} \boldsymbol{d}_1^\top & \boldsymbol{d}_2^\top & \cdots & \boldsymbol{d}_n^\top \end{array} \right)$$

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We look for $S \in \mathrm{GL}_k(\mathbb{F}_q)$ and $\boldsymbol{P} \in (\mathbb{F}_q^\times)^n \ltimes \mathfrak{S}_n$ such that

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Define

$$\mathscr{C}_1^{\mathrm{mat}} \stackrel{\mathsf{def}}{=} \mathrm{Span}_{\mathbb{F}_q}\left\{\boldsymbol{c}_i^\top \cdot \boldsymbol{c}_i\right\}, \quad \mathscr{C}_2^{\mathrm{mat}} \stackrel{\mathsf{def}}{=} \mathrm{Span}_{\mathbb{F}_q}\left\{\boldsymbol{d}_i^\top \cdot \boldsymbol{d}_i\right\}$$

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Fact

These matrix spaces are independent from $\boldsymbol{P}$! In addition:

$$\mathscr{C}_1^{mat} = \boldsymbol{S}\mathscr{C}_2^{mat}\boldsymbol{S}^\top.$$

Last observation

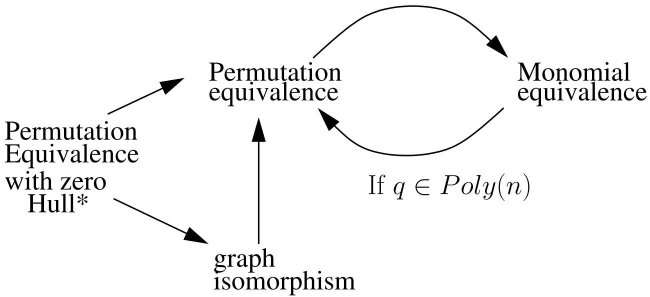
Remark

It might be possible that $\mathscr{C}_1^{mat}$ and $\mathscr{C}_2^{mat}$ are equivalent while $\mathscr{C}_1, \mathscr{C}_2$ are **not** *monomially equivalent. To address this issue, we consider slightly more complicated matrix codes:*

$$\mathscr{C}_1^{mat} \stackrel{def}{=} \mathrm{Span}\left\{ \begin{pmatrix} \boldsymbol{c}_i^\top \cdot \boldsymbol{c}_i \\ \hline \boldsymbol{M}_i \end{pmatrix} \right\},$$

where $\boldsymbol{M}_i \in \mathcal{M}_k(\mathbb{F}_q)$ is zero but at the i–th row which is all-one.

A picture


Permutation equivalence
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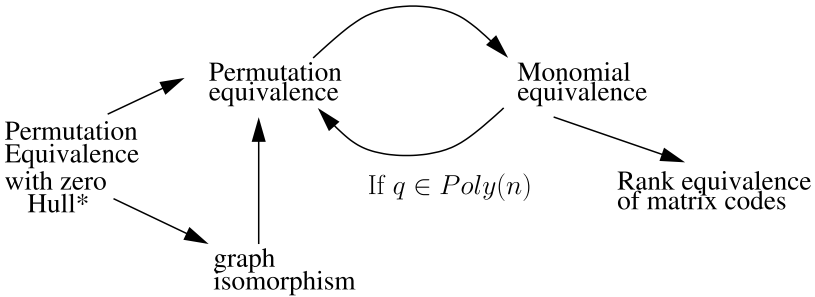
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Conclusion

- In Hamming metric
  - permutation equivalence is "most of the times" easy to solve;
  - monomimal equivalence is hard to solve.
- In rank metric
  - Equivalence of $\mathbb{F}_{q^m}$–linear codes would be easy even when hiding the $\mathbb{F}_{q^m}$–linear structure
  - Equivalence of non structured matrix codes is at least as hard (in the worst case) to monomial equivalence in Hamming metric.

Thank you for your attention!