Quantum Circuits and ZX-Calculus

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Advantages allowed by quantum computing:

- algorithms (Shor, Grover, ...)
- cryptography
- simulation

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Develop tools for :

- representing
- analysing/reasoning
- optimising
- verifying

quantum program/protocols.

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- Isolated systems evolve unitarily: $|\psi_1\rangle = U|\psi_0\rangle$ with $U^\dagger U = id = UU^\dagger$

- $H := \frac{1}{\sqrt{2}} \begin{array}{cc} & \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array}$ is unitary

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- $$\mathrm{QFT}_2 \circ \overbrace{|0+\rangle}^{\substack{\text{state preparation} \\ =|0\rangle\otimes|+\rangle}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \circ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{array}{cc} \begin{pmatrix} 2 \\ 1+i \\ 0 \\ 1-i \end{pmatrix} & \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$$

measurement $\rightarrow$ 50% $|00\rangle$, 25% $|01\rangle$, 25% $|11\rangle$

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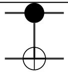
Quantum gates:

X or *Not*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

CX or *CNot*



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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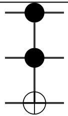

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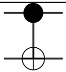

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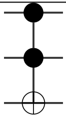
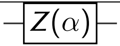


$$|a, b, c\rangle \mapsto |a, b, ab \oplus c\rangle$$

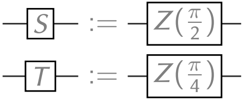
H

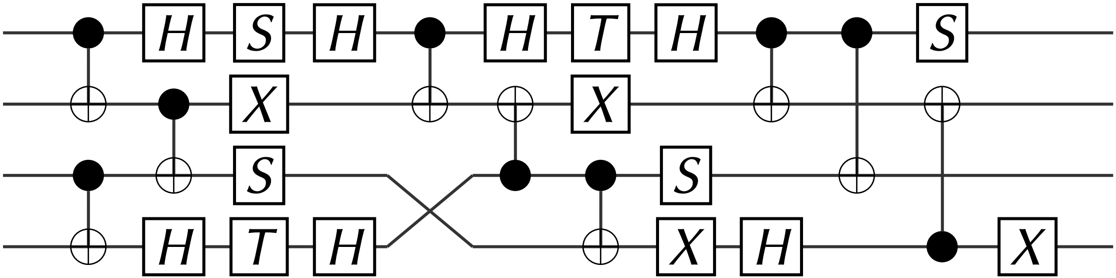
$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

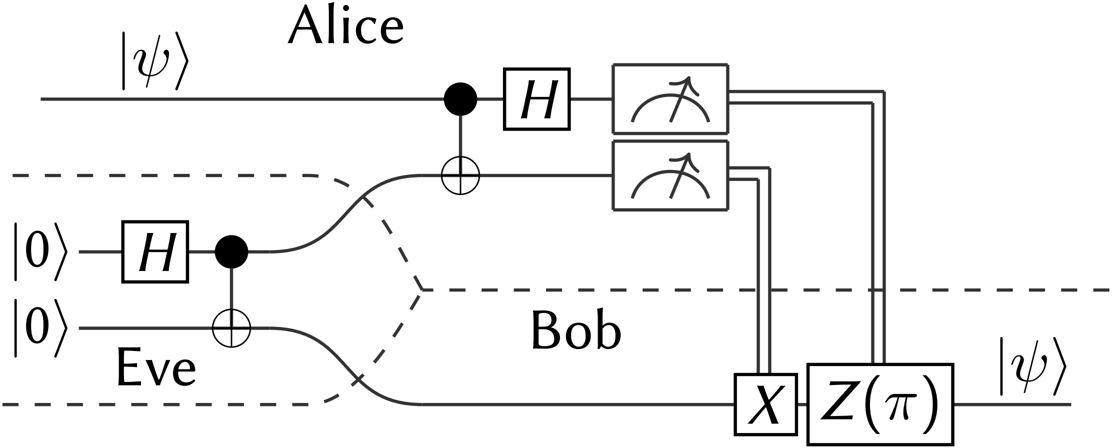
$Z(\alpha)$ or $R_Z(\alpha)$



$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$




H
S
H
H
T
H
S
X
X
S
S
H
T
H
X
H
X


Alice
$|\psi\rangle$
$|0\rangle$
$|0\rangle$
H
H
Eve
Bob
X
$Z(\pi)$
$|\psi\rangle$

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Any measurement can be ”pushed” to the very end of the procedure, without affecting the outcome.

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Usual scheme for quantum computing:

1. Initialise register of qubits
2. Apply unitary gates
3. Measure qubits

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Theorem : Universality[2]

The gate set $\{H, Z(\alpha), CX\}_{\alpha\in\mathbb{R}}$ is universal.

[2] [Barenco *et al.*'95]

[3] Gottesman-Knill theorem, [Gottesman'98]

[4] [Boykin, Mor, Pulver, Roychowdhury, Vatan' 00]

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- Clifford+T fragment : $\alpha \in \frac{\pi}{4}\mathbb{Z}$
  - approx. universal[4], with efficient approximation[5]

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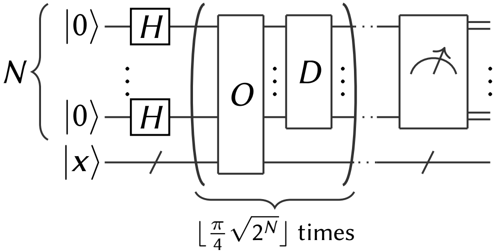

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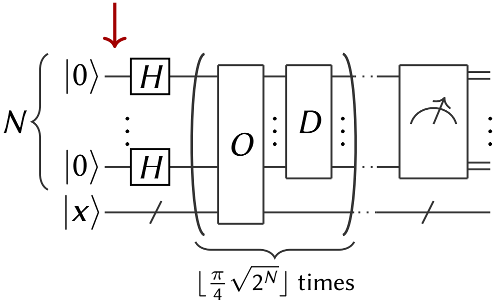

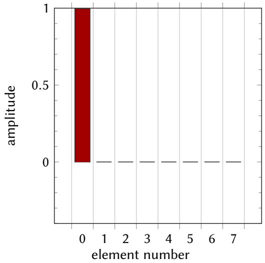

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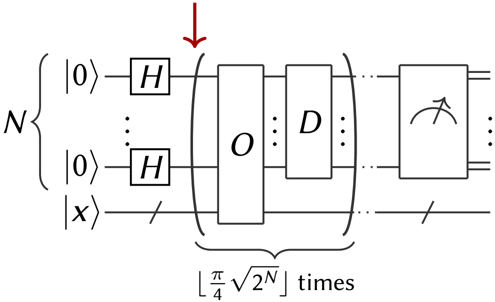

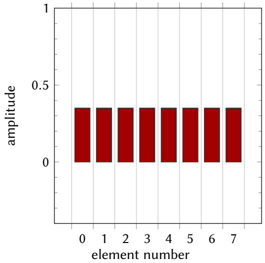

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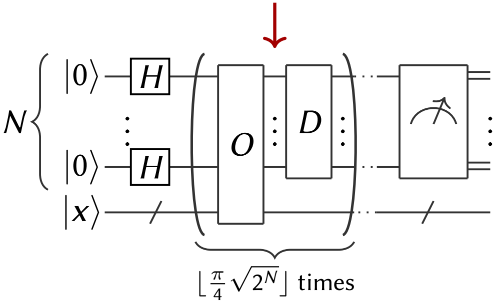

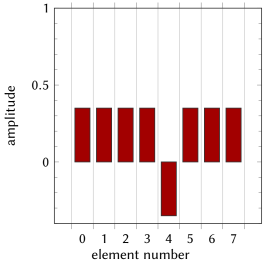

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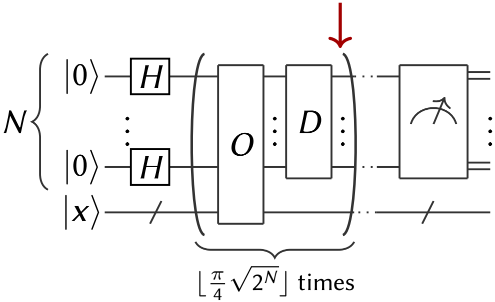

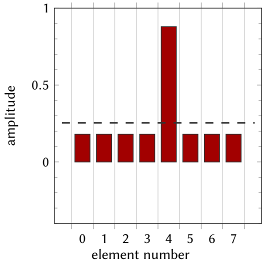

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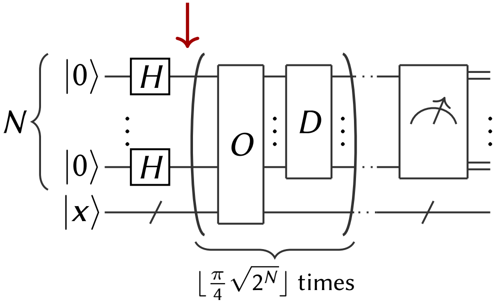

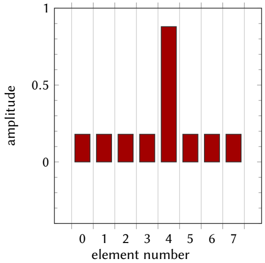

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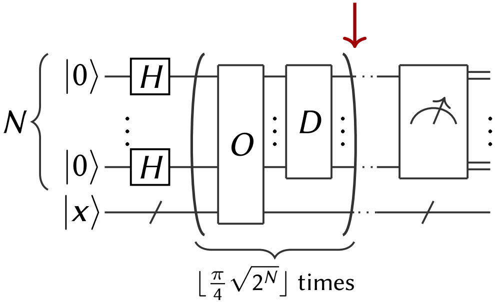

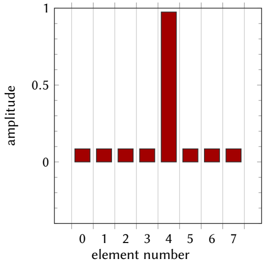

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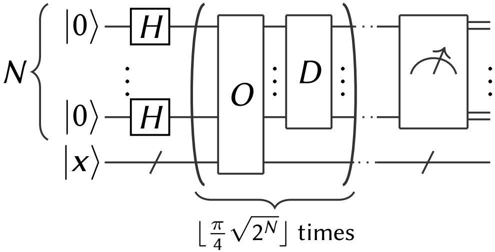

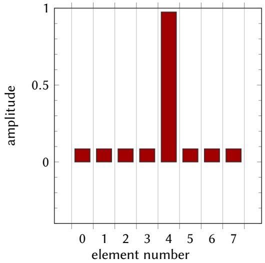

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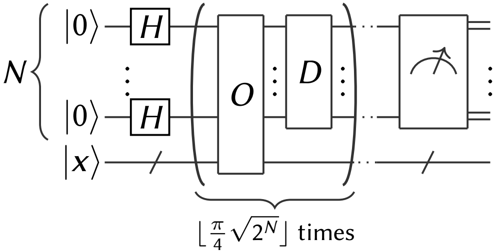

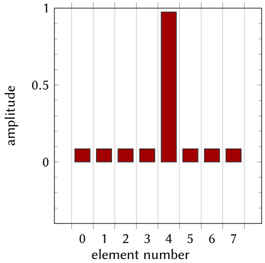

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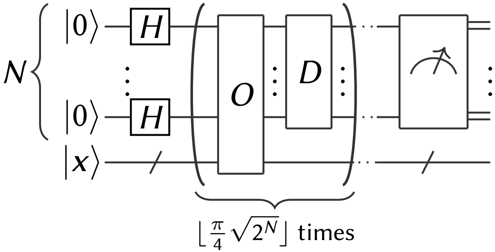

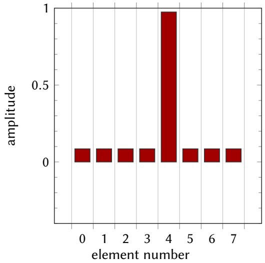

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Algo in $O(\sqrt{2^N})$ vs. $O(2^N)$ classically

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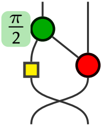
- Manipulates string diagrams e.g. 

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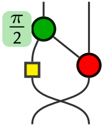
- Manipulates string diagrams e.g. 

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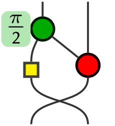

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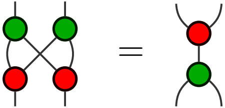

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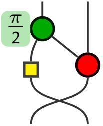
- Manipulates string diagrams e.g. 

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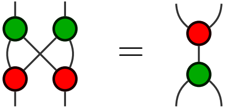
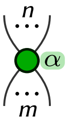
- Has a powerful equational theory e.g. 

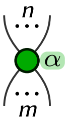
- Represents quantum circuits and more

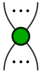
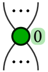
The Generators

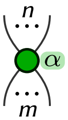
- A spider:  $::$

$$\begin{array}{cc} & \overbrace{\hphantom{1 \quad 0 \quad \cdots \quad \cdots \quad 0}}^{2^n} \\ 2^m \left\{ \vphantom{\begin{matrix}1\\0\\ \vdots\\0\\0\end{matrix}} \right. & \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix} \end{array}$$

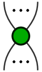
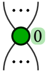
$= |0^m\rangle\langle 0^n| + e^{i\alpha}|1^m\rangle\langle 1^n|$

- A spider:  $::$ $$\begin{array}{cc} & \overbrace{\hphantom{\begin{matrix}1 & 0 & \cdots\cdots & 0\end{matrix}}}^{2^n} \\ 2^m\left\{\vphantom{\begin{matrix}1\\0\\ \vdots\\ \vdots \\ 0\end{matrix}}\right. & \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix} \end{array}$$ $$= |0^m\rangle\langle 0^n| + e^{i\alpha}|1^m\rangle\langle 1^n|$$

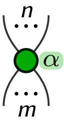
 $:=$ 

- A spider: 

$:: \; 2^m \left\{ \overbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix}}^{2^n} \right.$
$= |0^m\rangle\langle 0^n| + e^{i\alpha} |1^m\rangle\langle 1^n|$

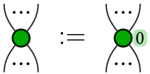
 $:=$ 

- A change of basis:  $:: \; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

- A spider: 

$$:: \; 2^m \left\{ \overbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha}\end{pmatrix}}^{2^n} \right.$$

$$= |0^m\rangle\langle 0^n| + e^{i\alpha}|1^m\rangle\langle 1^n|$$



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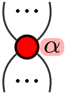
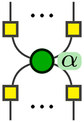
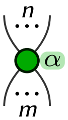
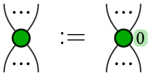
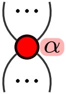
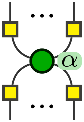
- Another spider:  $:=$  $:: \; |+^m\rangle\langle +^n| + e^{i\alpha}\,|-^m\rangle\langle -^n|$

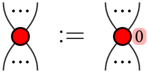
- A spider: 

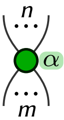
$$:: \; 2^m \left\{ \overbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha}\end{pmatrix}}^{2^n} \right.$$
$$= |0^m\rangle\langle 0^n| + e^{i\alpha}|1^m\rangle\langle 1^n|$$


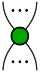
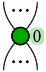
- A change of basis:  $:: \; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

- Another spider: 

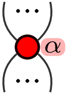
$:=$ 

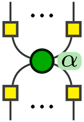
$:: \; |+^m\rangle\langle +^n| + e^{i\alpha}\,|-^m\rangle\langle -^n|$


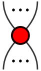
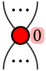
- A spider: 

$$:: \quad 2^m\left\{\overbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & e^{i\alpha} \end{pmatrix}}^{2^n}\right. \qquad = |0^m\rangle\langle 0^n| + e^{i\alpha}|1^m\rangle\langle 1^n|$$
 $:=$ 

- A change of basis:  $:: \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

- Another spider: 

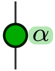
$:=$ 

$:: |+^m\rangle\langle +^n| + e^{i\alpha}\,|-^m\rangle\langle -^n|$
 $:=$ 

- Wires: $:: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,  $:: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,  $:: \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$,  $:: \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$

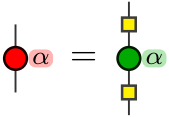
One-qubit Operators

$Z(\alpha)$:
rotation around Z



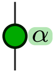
Hadamard:

$X(\alpha)$:
$= HZ(\alpha)H$



One-qubit Operators

$Z(\alpha)$:
rotation around Z



Hadamard: 

$X(\alpha)$:
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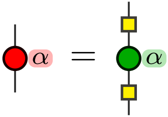
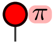


States and Projectors

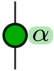
$\sqrt{2}\,|0\rangle$: 

$\sqrt{2}\,|1\rangle$: 

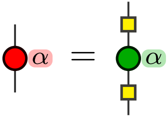
$|00\rangle + |11\rangle$: 

$\langle 00| + \langle 11|$: 

One-qubit Operators

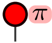
$Z(\alpha)$:
rotation around Z 

Hadamard: 

$X(\alpha)$:
$= HZ(\alpha)H$ 

States and Projectors

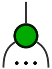
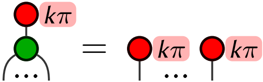
$\sqrt{2}\,|0\rangle$: 

$\sqrt{2}\,|1\rangle$: 

$|00\rangle + |11\rangle$: 

$\langle 00| + \langle 11|$: 

Green Spider

Copy:  s.t. 

One-qubit Operators

$Z(\alpha)$:
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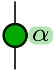

Hadamard:

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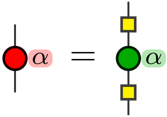
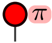


States and Projectors

$\sqrt{2}\,|0\rangle$:

$\sqrt{2}\,|1\rangle$:

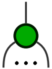


$|00\rangle + |11\rangle$:

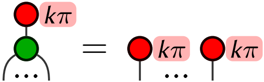
$\langle 00| + \langle 11|$:

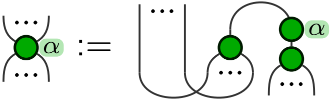
Green Spider

Copy:



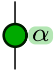
s.t.





One-qubit Operators

$Z(\alpha)$:
rotation around Z



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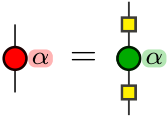
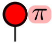


States and Projectors

$\sqrt{2}\,|0\rangle$:

$\sqrt{2}\,|1\rangle$:

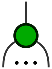


$|00\rangle + |11\rangle$:

$\langle 00| + \langle 11|$:

Green Spider

Copy:



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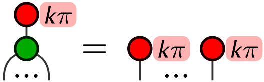
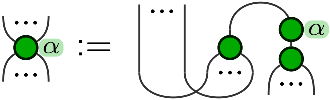
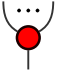




Red Spider

XOR:



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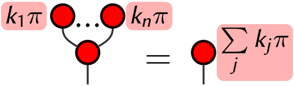
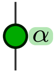


One-qubit Operators

$Z(\alpha)$:
rotation around Z



Hadamard:

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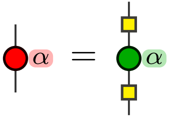
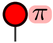


States and Projectors

$\sqrt{2}\,|0\rangle$:

$\sqrt{2}\,|1\rangle$:

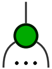


$|00\rangle + |11\rangle$:

$\langle 00| + \langle 11|$:

Green Spider

Copy:



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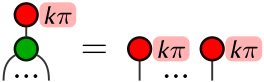
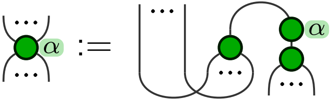
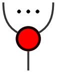




Red Spider

XOR:



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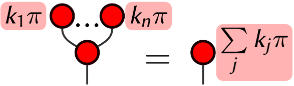
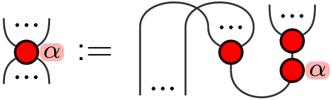


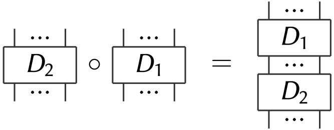


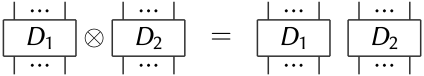
Generators

ZX-Calculus [Coecke,Duncan'08] in Short

Generators

Compositions





Generators

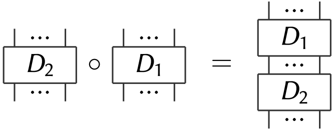
Compositions

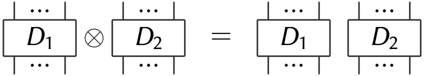
Standard Interpretation

$$[\![.]\!] : \mathbf{ZX} \to \mathcal{M}(\mathbb{C})$$

Generators

Compositions

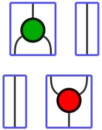




Standard Interpretation

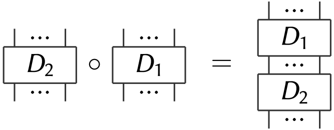
$$[\![.]\!] : \mathbf{ZX} \to \mathcal{M}(\mathbb{C})$$

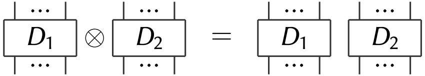
Example



Generators

Compositions





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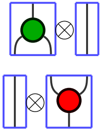
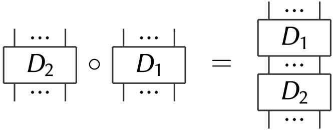
Example

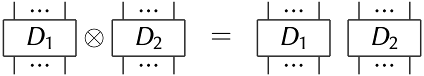


ZX-Calculus [Coecke,Duncan'08] in Short

Generators

Compositions





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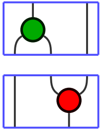
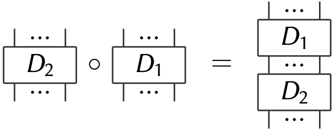
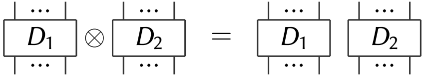


Generators

Compositions





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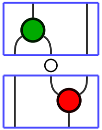
Example



ZX-Calculus [Coecke,Duncan'08] in Short

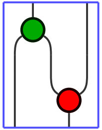
Generators

Compositions

Standard Interpretation

$$[\![.]\!] : \mathbf{ZX} \to \mathcal{M}(\mathbb{C})$$

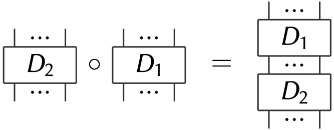
Example

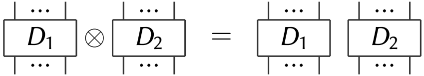


ZX-Calculus [Coecke,Duncan'08] in Short

Generators

Compositions





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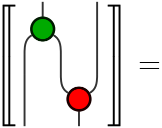
Example



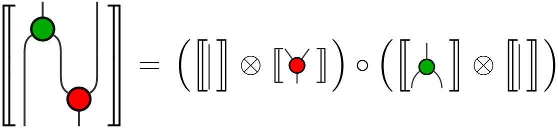
Generators

Compositions

Standard Interpretation

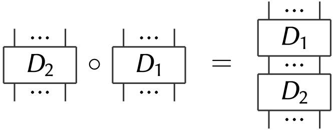
$$[\![.]\!] : \mathbf{ZX} \to \mathcal{M}(\mathbb{C})$$

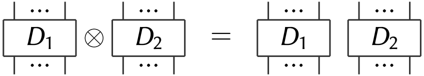
Example



Generators

Compositions





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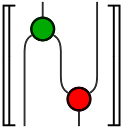
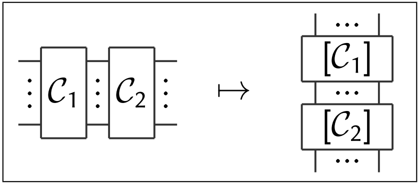
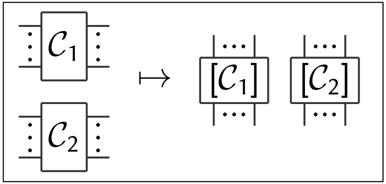
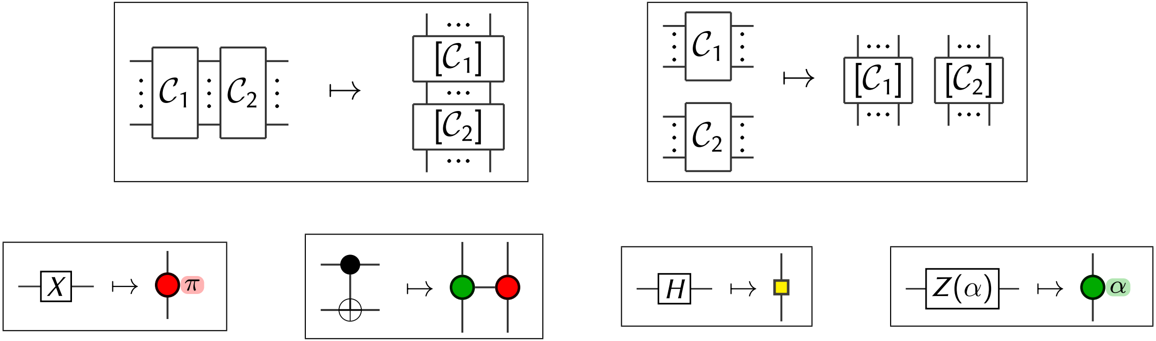
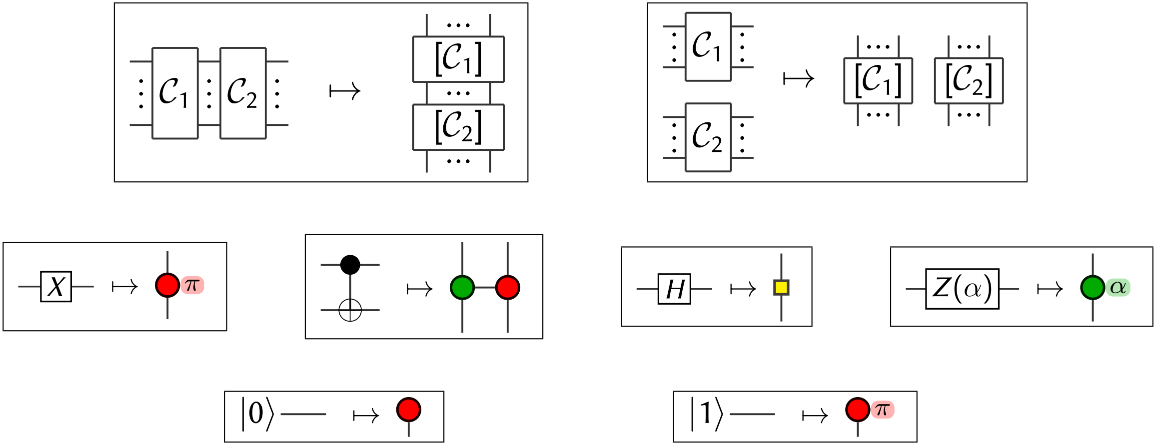


$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$


$\mathcal{C}_1$
$\mathcal{C}_2$
$\mapsto$
$[\mathcal{C}_1]$
$[\mathcal{C}_2]$


$\mathcal{C}_1$
$\mathcal{C}_2$
$\mapsto$
$[\mathcal{C}_1]$
$[\mathcal{C}_2]$


$\mathcal{C}_1$
$\mathcal{C}_2$
$\mapsto$
$[\mathcal{C}_1]$
$[\mathcal{C}_2]$
$\mathcal{C}_1$
$\mathcal{C}_2$
$\mapsto$
$[\mathcal{C}_1]$
$[\mathcal{C}_2]$
X
$\mapsto$
π
$\mapsto$
H
$\mapsto$
$Z(\alpha)$
$\mapsto$
α


C1
C2
[C1]
[C2]
X
π
H
Z(α)
α
|0⟩
|1⟩

Theorem (Universality)

We can represent any quantum operator using ZX-diagrams:

$$\forall f : \mathbb{C}^{2^n} \to \mathbb{C}^{2^m},\ \exists \underset{m}{\overset{n}{\boxed{D}}} \in \mathbf{ZX},\ \left\llbracket \underset{m}{\overset{n}{\boxed{D}}} \right\rrbracket = f$$

Theorem (Universality)

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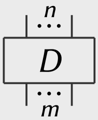
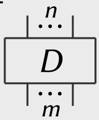
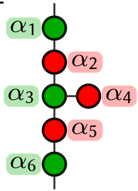
$\forall f : \mathbb{C}^{2^n} \to \mathbb{C}^{2^m},\ \exists$ 

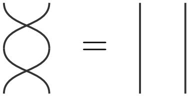
$\in \mathbf{ZX},\ \llbracket$ 

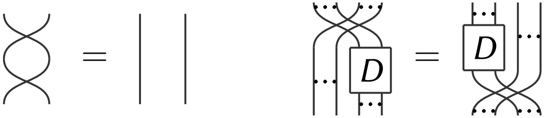
$\rrbracket = f$

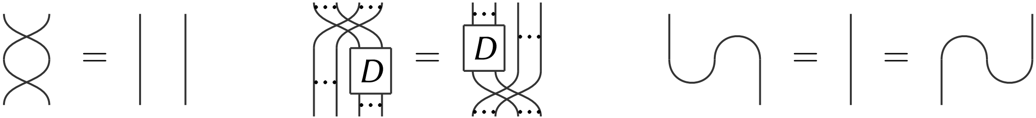
E.g. if $f : \mathbb{C}^2 \to \mathbb{C}^2,\ \exists \alpha_i,\ \llbracket$ 

$\rrbracket = f$

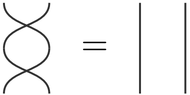
Equational Theory (the Backbone)

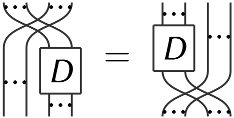


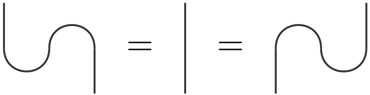

D
D

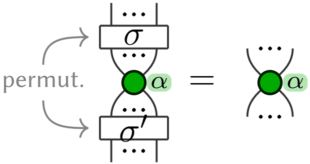

D
D

Equational Theory (the Backbone)

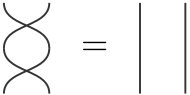


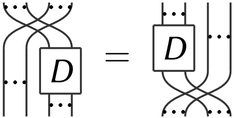


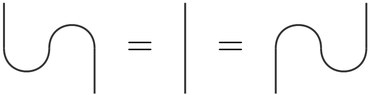


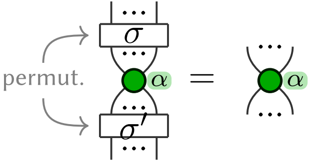


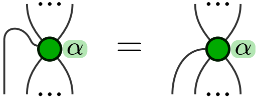
Equational Theory (the Backbone)



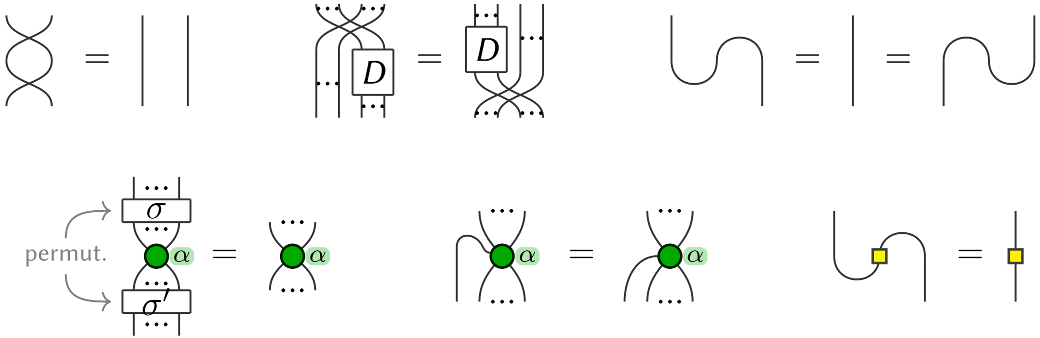




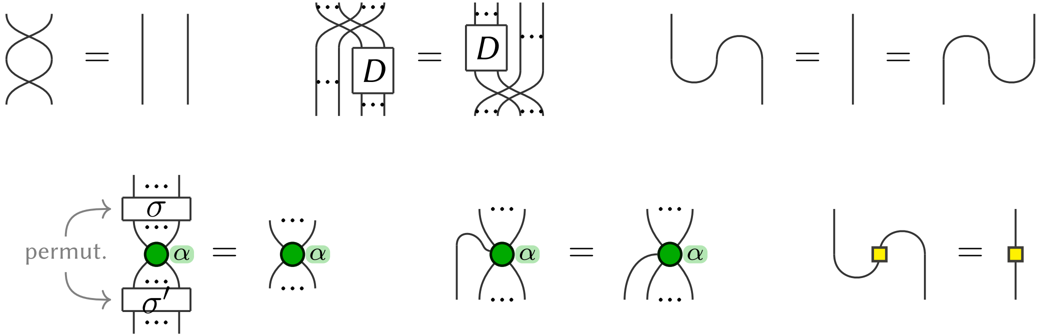




Equational Theory (the Backbone)

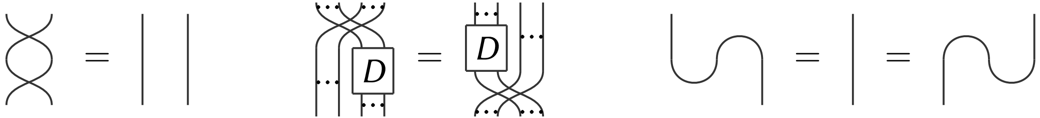


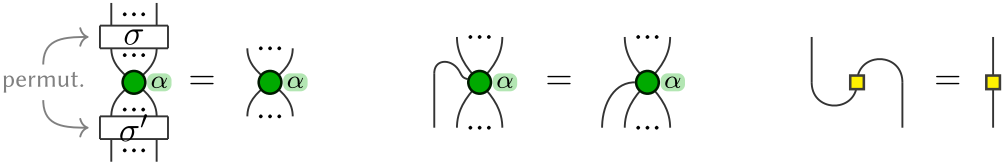
Equational Theory (the Backbone)



Only Connectivity Matters

ZX-diagrams can be seen as open graphs. Any graph isomorphism is a valid derivation in the equational theories.





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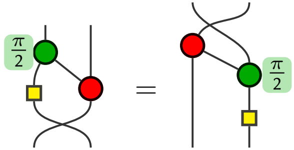
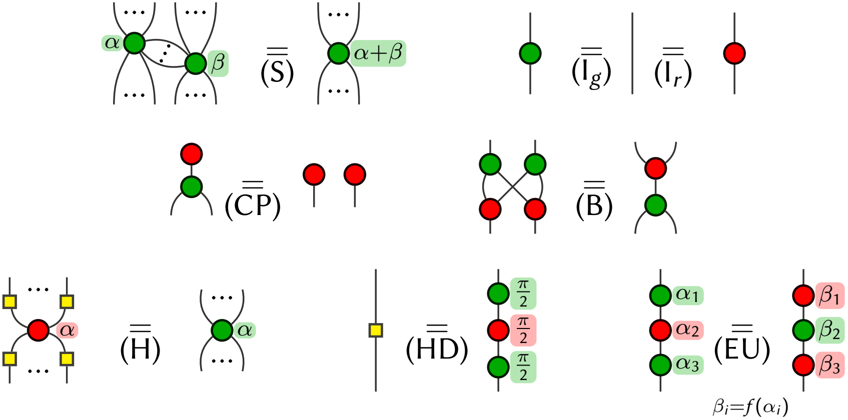
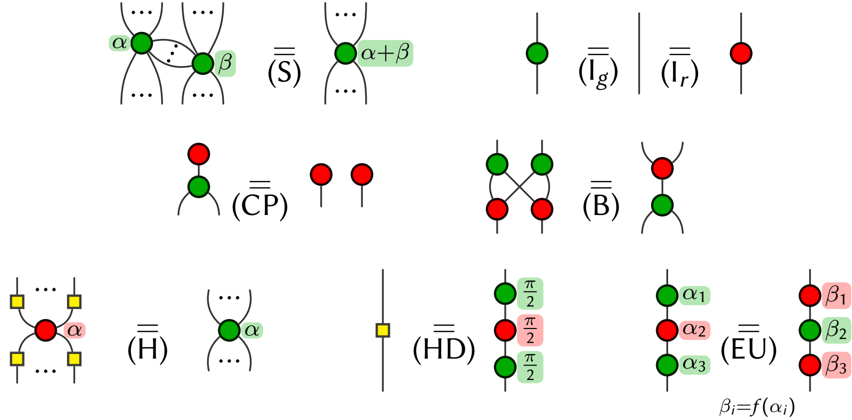
E.g.

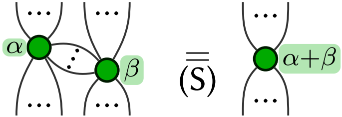


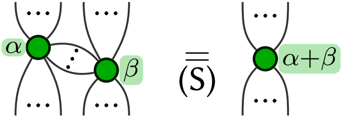
Equational Theory



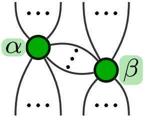


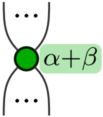
We write ZX $\vdash D_1 = D_2$. Every colour-swapped rule holds.


α
β
(S)
α+β



: comes from the study of †-Frobenius algebras in $\mathbb{C}^2$.



$\underset{(\mathrm{S})}{=}$


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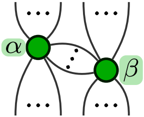
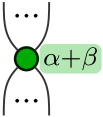
$\underset{(\mathrm{I}_g)}{=}$ | $\underset{(\mathrm{I}_r)}{=}$

$\underset{(S)}{=}$


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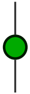
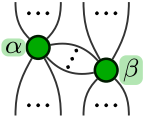
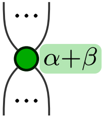
rotation of angle 0 $\rightarrow$  $\underset{(I_g)}{=}$ $\underset{(I_r)}{=}$  $\leftarrow$ rotation of angle 0

Understanding the Rules: the Spiders

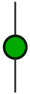


$\overset{(\mathrm{S})}{=}$



: comes from the study of †-Frobenius algebras in $\mathbb{C}^2$.

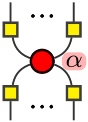
rotation of angle 0 $\rightarrow$



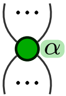
$\overset{(\mathrm{I}_g)}{=}$

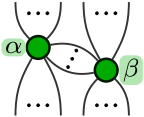
$\overset{(\mathrm{I}_r)}{=}$

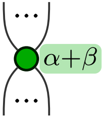
$\leftarrow$ rotation of angle 0



$\overset{(\mathrm{H})}{=}$





$\overset{(\mathrm{S})}{=}$


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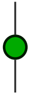
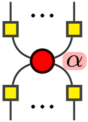
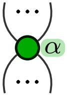
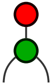
rotation of angle 0 $\to$

$\overset{(\mathrm{I}_g)}{=}$ | $\overset{(\mathrm{I}_r)}{=}$

$\leftarrow$ rotation of angle 0

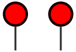


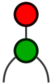
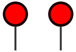
$\overset{(\mathrm{H})}{=}$


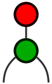
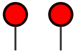
: change of basis

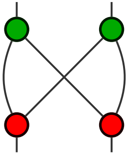
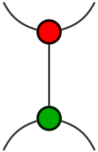


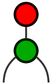
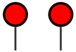
$\overset{(CP)}{=}$

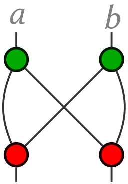


 $\overset{}{\underset{\text{(CP)}}{=}}$  : copy of classical 0.

 $\overset{=}{\text{(CP)}}$  : copy of classical 0.

 $\overset{=}{\text{(B)}}$ 

 $\underset{(CP)}{=}$  : copy of classical 0.



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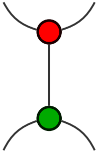

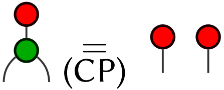

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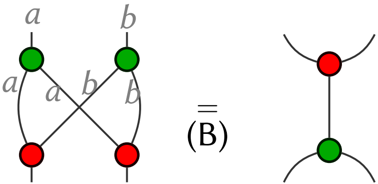

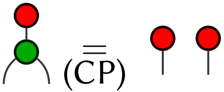

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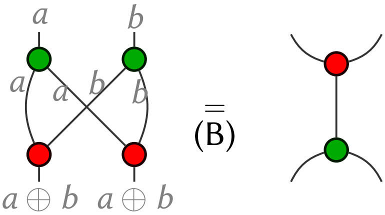

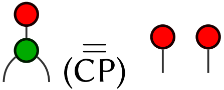

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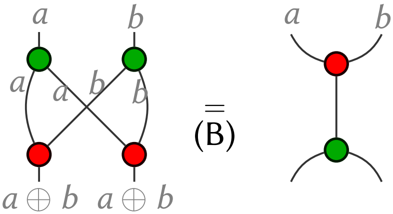

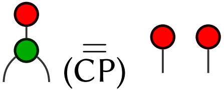

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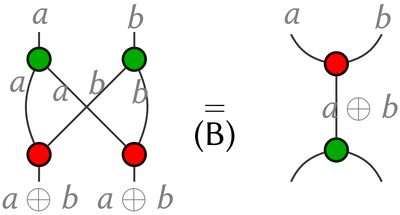

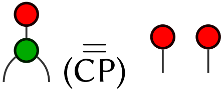

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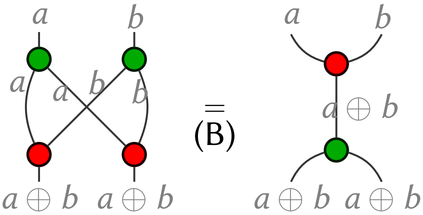

Rotations in $\mathbb{R}^3$:

$$\forall\theta,\ \exists\alpha_i,\quad R_?(\theta) = R_x(\alpha_3)\circ R_y(\alpha_2)\circ R_x(\alpha_1)$$

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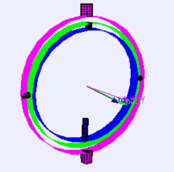

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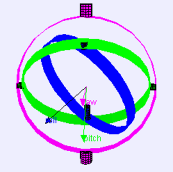

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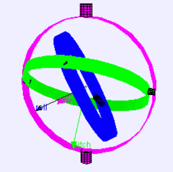

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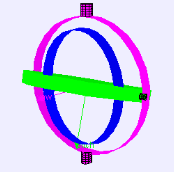

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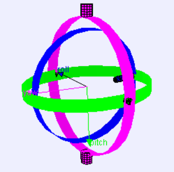

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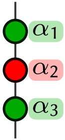

Understanding the Rules: Euler Angles

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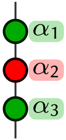

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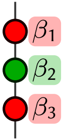

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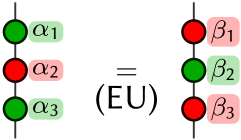

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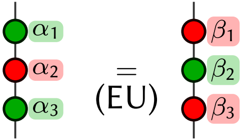

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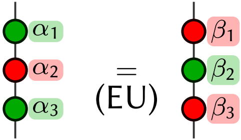

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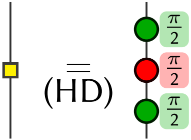
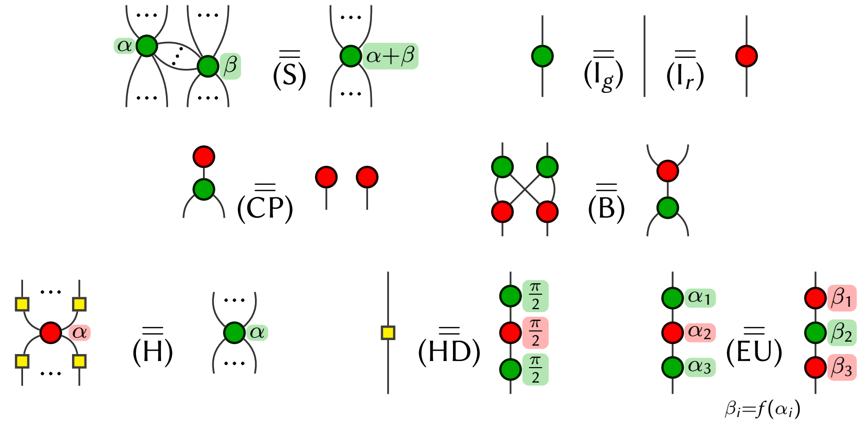
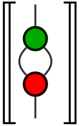
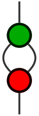
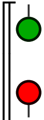


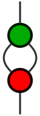
Equational Theory

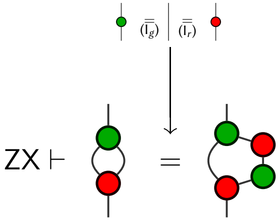




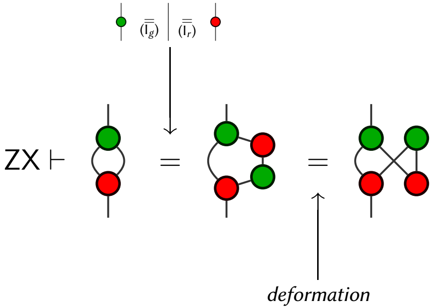
$$= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$\llbracket$  $\rrbracket = \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} \circ \begin{pmatrix}1 & 1\end{pmatrix} = \llbracket$  $\rrbracket$

ZX ⊢ 


ZX ⊢
$(\overline{\overline{\mathrm{I}}}_g)$
$(\overline{\overline{\mathrm{I}}}_r)$
=

Using the Rules: the Hopf Law



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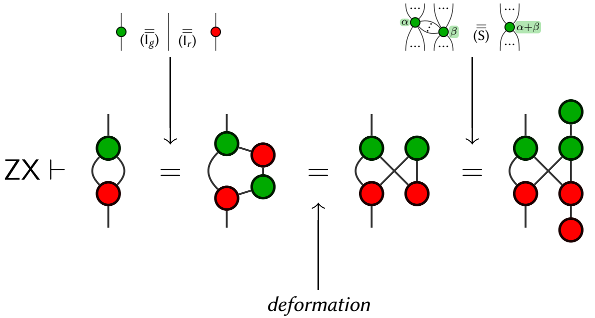
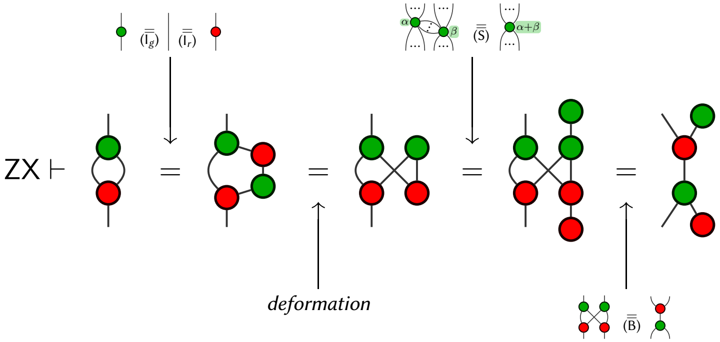
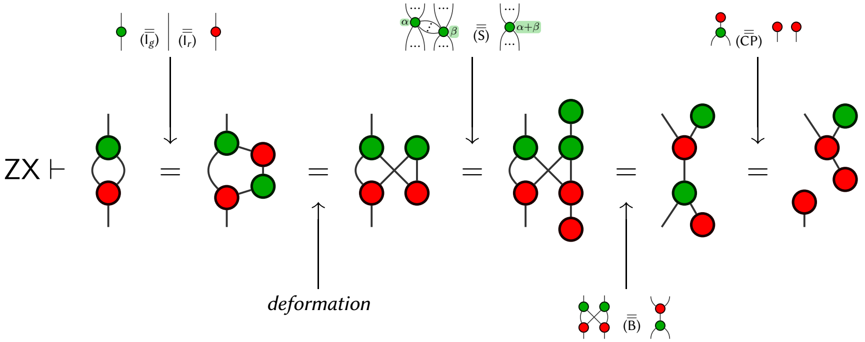



ZX ⊢
=
=
=
=
$(\overline{\overline{\mathrm{I}_g}})$
$(\overline{\overline{\mathrm{I}_r}})$
α
β
$(\overline{\overline{\mathrm{S}}})$
$\alpha+\beta$
deformation
$(\overline{\overline{\mathrm{B}}})$



ZX ⊢
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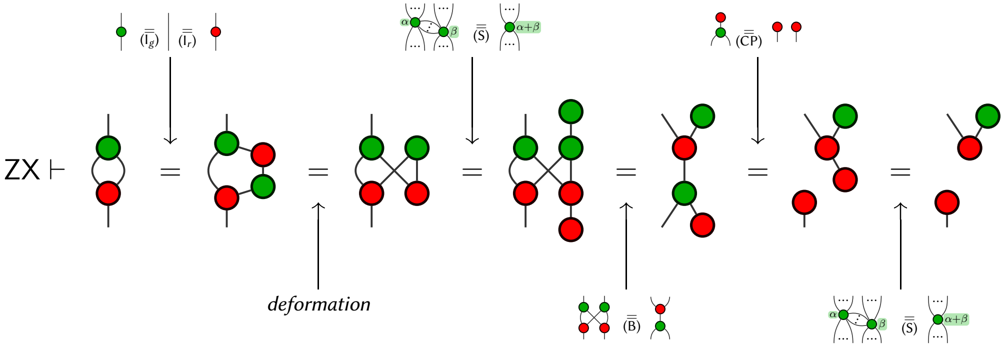

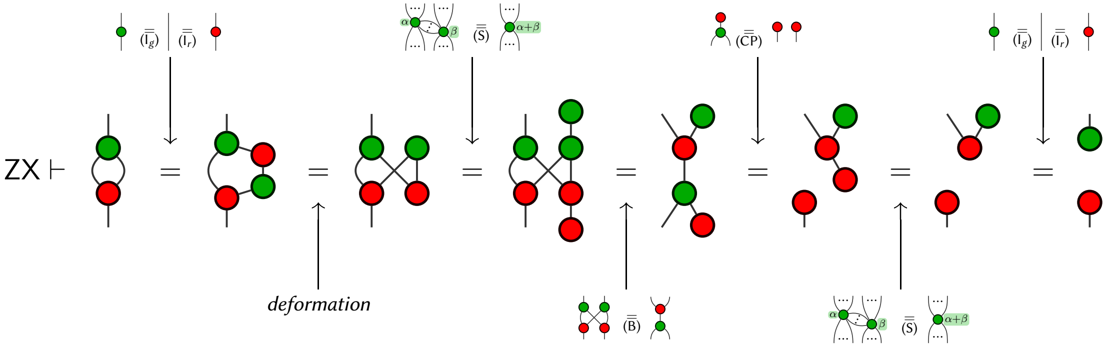
ZX ⊢
$\overset{=}{(\mathrm{I}_g)}$
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deformation
α
β
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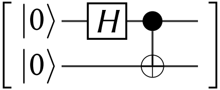
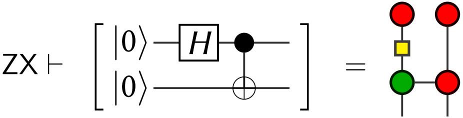
The EPR state: $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$

ZX $\vdash$



Using the Rules: The EPR Pair

The EPR state: $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$



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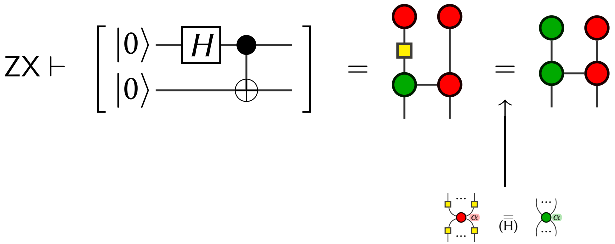

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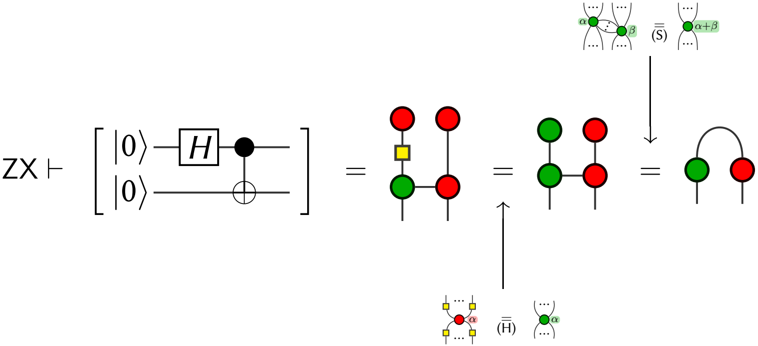

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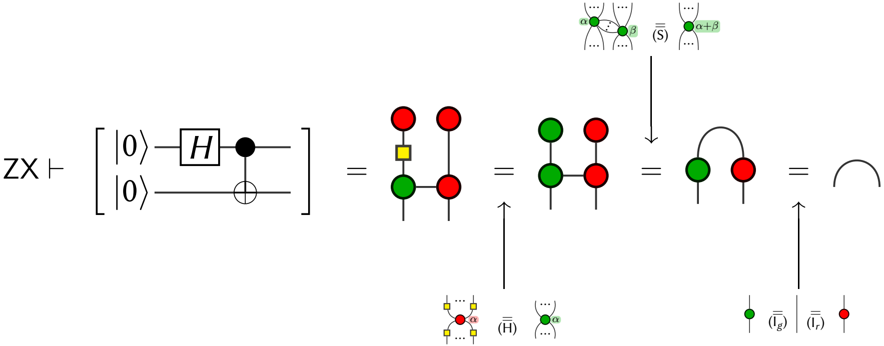

Theorem [V.'19]

The language is *complete*:

$$\forall D_1, D_2 \in \mathbf{ZX},\ [\![D_1]\!] = [\![D_2]\!] \iff \mathrm{ZX} \vdash D_1 = D_2$$

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Previous/other completeness results:

- $\frac{\pi}{2}$-fragment [Backens'14]
- π-fragment [Duncan,Perdrix'14]
- 1-qubit $\frac{\pi}{4}$-fragment [Backens'14]
- $\frac{\pi}{4}$-fragment [Jeandel,Perdrix,V.'18]
- full ZX (modified) [Hadzihasanovic,Ng,Wang'18]
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$$\Lambda|\psi\rangle := \sqrt{2}^n\,|+^n\rangle\langle 0| + |\psi\rangle\langle 1| = \begin{pmatrix} 1 & \psi_0 \\ \vdots & \vdots \\ 1 & \psi_{2^n-1} \end{pmatrix} \text{ i.e.:}$$

Sketch of (a) proof: Normal Forms

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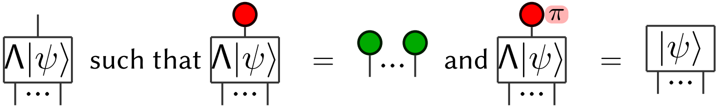

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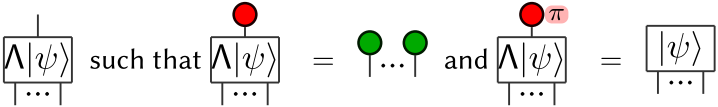
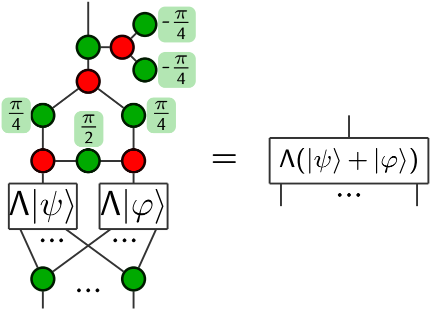
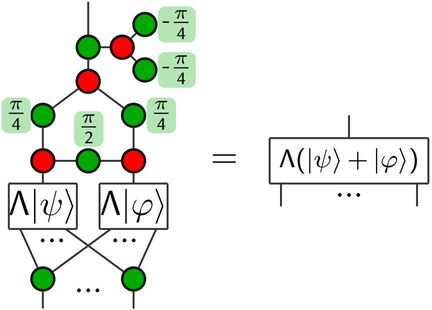
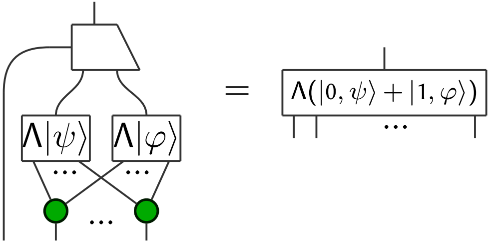


- Base case: controlled scalar:

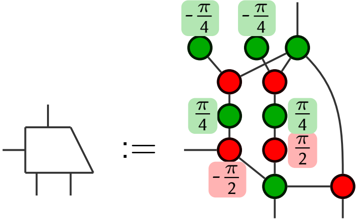
$$\Lambda x = \langle 0| + x\,\langle 1| = \begin{pmatrix} 1 & x \end{pmatrix}$$

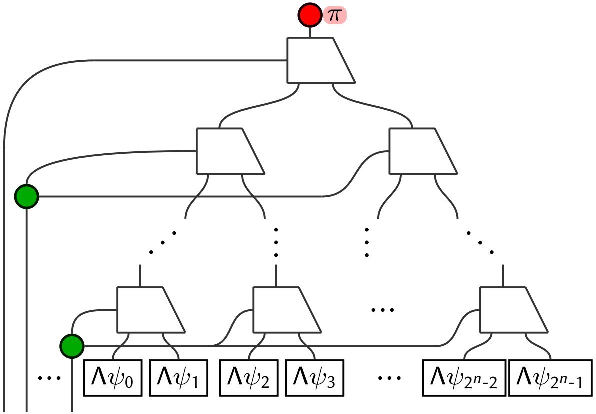

$-\frac{\pi}{4}$
$-\frac{\pi}{4}$
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{\pi}{4}$
$\Lambda|\psi\rangle$
$\Lambda|\varphi\rangle$
$=$
$\Lambda(|\psi\rangle + |\varphi\rangle)$





with





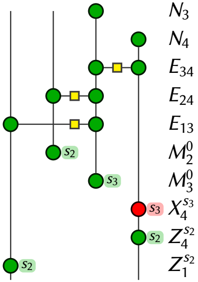
- Generators can be put in NF
- Compositions of states in NF can be put in NF
- Completeness on controlled scalars

$\Downarrow$

Completeness!

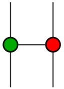
Plan

Unification



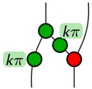
measure-ment calculus

$=$



quantum circuit

$=$



lattice surgery (surface codes)

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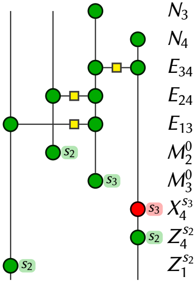

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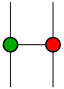
$=$ 

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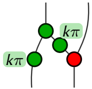
$=$ 

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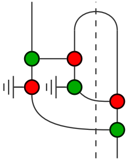
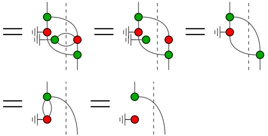
Verification



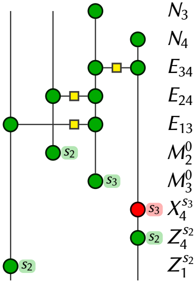
protocol



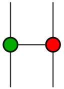
$=$ 

spec.

Unification



measure-ment calculus

$=$ 

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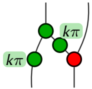
$=$ 

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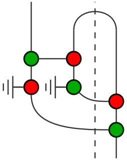

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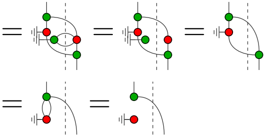
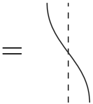
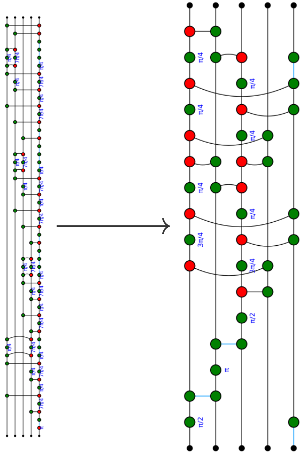


$=$ 

spec.

Optimisation



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- Used for verification (Quantomatic)