The Fundamental Theorem of Tropical Partial Differential Algebraic Geometry

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Introduction

Purpose of the presentation : present the Fundamental Theorem of Tropical (Partial) Differential Algebraic Geometry.

This work is at the intersection of differential algebra and tropical geometry.

Basic question : Given a ODE or PDE over a field k of characteristic zero, we want to have informations about the solutions (in the formal series ring $k[[t]]$) of this differential equation.

Basic question : Given a ODE or PDE over a field k of characteristic zero, we want to have informations about the solutions (in the formal series ring $k[[t]]$) of this differential equation.

It is known that for PDE such solutions can be impossible to compute.

[DL84] thm 4.11
There does not exist an algorithm to decide whether a linear PDE has a power series solution in $\mathbf{C}[[t_1, \cdots, t_r]]$ (for r large enough, say $r \geq 9$).

Differential algebra

The basic idea of differential algebra is the following association

Unknown function y and its derivatives <–> variables of a certain polynomial ring.

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The ring we are going to work with is the following one :

$$k[[t_1, \cdots, t_m]]\{y_1, \cdots, y_n\} = k[[t_1, \cdots, t_m]][y_{i,J} \mid 1 \leq i \leq n, J \in \mathbf{N}^m]$$

where $k[[t_1, \cdots, t_m]]$ is to understand as a coefficient ring.

This ring is called the ring of differential polynomials.

Examples

1. If $F = y' - y$ is an ODE with coefficients in k, the associated differential polynomial is $y_1 - y_0$.
2. If $F = t_1 \frac{\partial^2 y}{\partial t_1 \partial t_2} + t_2 \frac{\partial y}{\partial t_2} + y$ with coefficients in $k[[t_1, t_2]]$ then the associated differential polynomial is $t_1 y_{1,1} + t_2 y_{0,1} + y_{0,0}$.

We endow the ring $k[[t_1, \cdots, t_m]]\{y_1, \cdots, y_n\}$ with one or many derivations.

Derivations

Let $(R, +, \cdot)$ be a ring. We say that $\delta\colon R \to R$ is a derivation is δ is linear for $+$ and verifies

$$\delta(ab) = a\delta(b) + \delta(a)b$$

for $a, b \in R$.

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Examples

1. The zero derivation on R

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Examples

1. The zero derivation on R
2. On $k[[t_1, \cdots, t_m]]$ the partial derivation ∂_{t_i} with respect to t_i.

Let $m \in \mathbf{N}$. We consider the ring

$$R_{m,n} = k[[t_1, \cdots, t_m]][y_{i,J} \mid 1 \leq i \leq n, J \in \mathbf{N}^m]$$

which can be denoted $k[[t_1, \cdots, t_m]]\{y_1, \cdots, y_n\}$.

Derivations on $R_{m,n}$

Let's define $D = \{\delta_1, \cdots, \delta_m\}$ derivations on $R_{m,n}$ such that:

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Let's define $D = \{\delta_1, \cdots, \delta_m\}$ derivations on $R_{m,n}$ such that:

1. δ_i extend ∂_{t_i}
2. $\delta_i(y_{j,J}) = y_{j,J+e_i}$ where e_i stands for $(0, \cdots, 1, \cdots, 0)$ where the 1 is at the ith position.

More generally if $I = (i_1, \cdots, i_m) \in \mathbf{N}^m$, we denote

$$D(I)(y_{i,J}) := \delta_1^{i_1} \cdots \delta_m^{i_m}(y_{i,J}) = y_{i,J+I}$$

Study object : the differential ring

$$R_{m,n} = k[[t_1, \cdots, t_m]][y_{i,J} \mid 1 \leq i \leq n, J \in \mathbf{N}^m]$$

with the derivations $D = \{\delta_1, \cdots, \delta_m\}$.

Let k be a field of characteristic zero. Let $R_{m,n}$ with m derivations $D = \{\delta_1, \cdots, \delta_m\}$ previously defined.

Let I be an (algebraic) ideal of $R_{m,n}$.

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Let I be an (algebraic) ideal of $R_{m,n}$.

We say that I is a differential ideal if for every $a \in I$ and for every $1 \leq i \leq n$ we have $\delta_i(a) \in I$.

Let

$$F \in k[[t_1, \cdots, t_m]][y_{i,J} \mid 1 \leq i \leq n, J \in \mathbf{N}^m]$$

We say that $\gamma = (\gamma_1, \cdots, \gamma_n) \in k[[t_1, \cdots, t_m]]^n$ is a solution of F if

$$F((D(J)(\gamma_i)_{1 \leq i \leq n, J \in \mathbf{N}^m}) = 0.$$

We will denote it $F(\gamma) = 0$.

Let $I \subset k[[t_1, \cdots, t_m]][y_{i,J} \mid 1 \leq i \leq n, J \in \mathbf{N}^m]$ be a differential ideal. We denote by $\mathrm{Sol}(I)$ the set

$$\{\gamma \in k[[t_1, \cdots, t_m]]^n \mid \forall F \in I, F(\gamma) = 0\}.$$

The fundamental theorem

The fundamental theorem of tropical differential algebraic geometry is originally a question that arises in an article of D. Grigoriev ([Gri17])

The theorem has been proven in the ordinary case (only 1 derivative) in [AGT16] by F. Aroca, C. Garay-López & Z. Toghani.

Then 4 years later the generalization for PDE was proven in [FGLH$^+$20].

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In this section we are first going to state the theorem for the ordinary case.

As said in the introduction the solutions of a differential system can be hard if not impossible to compute.

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Let $\gamma = \sum_{i \geq 0} a_i t^i \in k[[t]]$. We call the support of γ the subset of $\mathcal{P}(\mathbf{N})$

$$\mathrm{Supp}(\gamma) = \{i \in \mathbf{N} \mid a_i \neq 0\}.$$

Example

If $\gamma = t^5 + t + 3$ then $\mathrm{Supp}(\gamma) = \{5, 1, 0\}$.

If I is a differential ideal of $k[[t]]\{y\}$, we denote

$$\mathrm{Supp}(\mathrm{Sol}(I)) = \{\mathrm{Supp}(\gamma) \mid \gamma \in \mathrm{Sol}(I)\}.$$

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$$\mathrm{Supp}(\mathrm{Sol}(I)) = \{\mathrm{Supp}(\gamma) \mid \gamma \in \mathrm{Sol}(I)\}.$$

The goal is to understand $\mathrm{Supp}(\mathrm{Sol}(I))$ by looking at a certain system of tropical equations.

Let $F = \sum_{m \in \mathbf{N}} a_m E_m \in k[[t]]\{y\}$ with $a_m \in k[[t]]$ and E_m a monomial in y and its derivative. More precisely

$$E_m = \prod_{i \in \mathbf{N}} y_i^{m_i}.$$

Then the tropicalization of F is

$$\operatorname{trop}(F) = \sum_{m \in \mathbf{N}} \min \operatorname{Supp}(a_m) E_m.$$

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Then the tropicalization of F is

$$\operatorname{trop}(F) = \sum_{m \in \mathbf{N}} \min \operatorname{Supp}(a_m) E_m.$$

Here the tropicalization is a formal operation. Only the set where the coefficients of F live changes.

A tropical polynomial $\mathrm{trop}(F)$ can be seen as an evaluation map

$$\mathrm{trop}(F)\colon \mathcal{P}(\mathbf{N}) \to \mathcal{P}(\mathbf{N}).$$

The operations $+$ and $\cdot$ are replace in $\mathcal{P}(\mathbf{N})$ by min and $+$.

More precisely for $S \in \mathcal{P}(\mathbf{N})$ we define

$$y_i(S) = \min\{k - i \mid k \in S, k - i \geq 0\} = \min(S - i)_+$$

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Example

If $S = \{0, 1, 5, 8, 9\}$ then $y_3(S) = \min\{-3, -2, 2, 5, 6\}_+ = 2$

Tropical system

If $F = \sum_{m \in \mathbf{N}} a_m E_m \in k[[t]]\{y\}$ with $E_m = \prod_{i \in \mathbf{N}} y_i^{m_i}$ and $S \in \mathcal{P}(\mathbf{N})$ then

$$E_m(S) = \min \sum_{i \in \mathbf{N}} m_i \cdot y_i(S)$$

and

$$\operatorname{trop}(F)(S) = \min_{m \in \mathbf{N}} \{\min \operatorname{Supp}(a_m) + E_m(S)\}$$

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Example
If $S = \{0, 1, 2, 4\}$ then
$(y_1^2 - t y_0)(S) = \min(2 \min\{-1, 0, 1, 3\}_+, 1 + \min\{0, 1, 2, 4\}) = 0.$

Let $E_m = \prod_{i \in \mathbf{N}} y_i^{m_i}$ and $S \in \mathcal{P}(\mathbf{N})$. Let $\varphi \in k[[t]]$ be a formal serie of support S.

Property

$$\operatorname{trop}(E_m)(S) = \operatorname{Supp}(E_m(\varphi)).$$

Remark

If $F \in k[[t]]\{y\}$, in general

$$\operatorname{trop}(F)(S) \neq \operatorname{Supp}(F(\varphi)).$$

Example

If $F = y_1 - y_0$, $S = \{0, 1, 2, \cdots\}$ and $\varphi = e^t$ then

$$\operatorname{trop}(F)(S) = 0$$

but

$$\operatorname{Supp}(F(\varphi)) = \operatorname{Supp}(0) = \emptyset.$$

Solution of a tropical equation

Let $F = \sum_{m \in \mathbf{N}} a_m E_m \in k[[t]]\{y\}$ and $S \in \mathcal{P}(\mathbf{N})$. We say that S is a solution of $\mathrm{trop}(F)$ if $\mathrm{trop}(F)(S) = \emptyset$ or if there exists $m_1 \neq m_2 \in \mathbf{N}$ such that

$$\mathrm{trop}(F)(S) = \mathrm{trop}(a_{m_1} E_{m_1})(S) = \mathrm{trop}(a_{m_2} E_{m_2})(S).$$

To sum up : S is a solution if the min is obtained at least twice (or comes from at least two different monomials).

Examples

1. If $F = y_1 - y_0$, $S = \{0, 1, 2, \cdots\}$ then

$$\mathrm{trop}(F)(S) = \min(0, 0) = 0$$

   and S is a solution of F.
2. If $F = y_3$ and $S = \{0, 1\}$ then $\mathrm{trop}(F)(S) = \emptyset$ and S is a solution of F.

Let's denote $\mathbb{T}$ the semi-ring $(\mathbf{N}, \min, +)$. And let's $\varphi \in k[[t]]$ of support S

$$\begin{array}{ccc} k[[t]]\{y\} & \xrightarrow{\text{trop}} & \mathbb{T}\{y\} \\ \Big\downarrow{\scriptstyle \mathrm{ev}_\varphi} & & \Big\downarrow{\scriptstyle \mathrm{ev}_S} \\ k[[t]] & \xrightarrow[\min \mathrm{Supp}]{} & \mathbb{T} \end{array}$$

Warning : this diagram is, in general, not commutative

Let k be an uncountable field, algebraically closed and of characteristic zero. Let I be a differential ideal of $k[[t]]\{y\}$.

Aroca, Garay-López, Toghani

The fundamental theorem of tropical differential algebraic geometry is the following statement:

$$\mathrm{Supp}(\mathrm{Sol}(I)) = \mathrm{Sol}(\mathrm{trop}(I))$$

Sketch of the proof:

• The inclusion $\mathrm{Supp}(\mathrm{Sol}(I)) \subset \mathrm{Sol}(\mathrm{trop}(I))$ is the "easy direction".

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• The inclusion $\mathrm{Supp}(\mathrm{Sol}(I)) \subset \mathrm{Sol}(\mathrm{trop}(I))$ is the "easy direction".

Let $\varphi \in \mathrm{Sol}(I)$ and $F \in I$ with $F = \sum_{m\in\mathbf{N}} a_m E_m$. Let's consider $a_{m_1}E_{m_1}$ such that $\min(\mathrm{Supp}(a_{m_1}E_{m_1}(\varphi))) = \min_{m\in\mathbf{N}}(\min(\mathrm{Supp}(a_m E_m(\varphi))))$. Since $F(\varphi) = 0$ there exists $m_2 \neq m_1 \in \mathbf{N}$ such that

$$\min(\mathrm{Supp}(a_{m_1}E_{m_1}(\varphi))) = \min(\mathrm{Supp}(a_{m_2}E_{m_2}(\varphi)))$$

Other direction #1:

• Thanks to the Taylor's formula, it can proven that finding a solution of I is the same as finding a solution of a certain J_I ideal of $k[y_0, y_1, \cdots]$ in $k^{\mathbf{N}}$.

• Let S be a support. Consider

$$V_{S,m} = \{(x_i)_{1 \leq i \leq m} \in k^m \mid x_i = 0 \text{ if and only if } i \notin S\}.$$

Then $\mathrm{Sol}(J_I) \cap V_{S,\infty}$ are in bijection with the solutions of I of support S.

Other direction #2:

• If $\mathrm{Sol}(J_I) \cap V_{S,\infty}$ is empty, then there exists some integer m and ideal $J_{I,m}$ of $k[y_0, \cdots, y_m]$ such that $\mathrm{Sol}(J_{I,m}) \cap V_{S,m}$ is empty.

• Assume that there exists no solution of I of support S. Thanks to some arguments (the Nullstenllensatz for example) it's possible to construct an element $F \in I$ such that S is not a solution of $\mathrm{trop}(F)$.

Partial fundamental theorem

The goal is to generalize the fundamental theorem to PDE.

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Difficulty : The major difficulty of this generalization was to find adequate definitions for notions of tropicalization of a partial differential polynomial and evaluation of these tropicalizations.

As we have seen before the tropicalization of a certain $F=\sum_{m\in\mathbf{N}} a_m E_m \in k[[t]]\{y\}$ consists of

$$\mathrm{trop}(F)=\sum_{m\in\mathbf{N}} \min(\mathrm{Supp}(a_m))E_m.$$

If $a\in k[[t_1,\cdots,t_m]]$ is a formal series in m variables then $\mathrm{Supp}(a)\in\mathcal{P}(\mathbf{N}^m)$. In $\mathcal{P}(\mathbf{N}^m)$ the notion of min can be generalized in different ways.

Let $\gamma = \sum_{(j_1,\cdots,j_m)\in\mathbf{N}^m} a_{(j_1,\cdots,j_m)} t_1^{j_1}\cdots t_m^{j_m} = \sum_{J\in\mathbf{N}^m} a_J T^J$, we define the

$$\mathrm{Supp}(\gamma) = \{J \in \mathbf{N}^m \mid a_J \neq 0\}.$$

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Example

Si $\gamma = t_1^3 t_2^2 + t_1^4 + t_2^2$ alors $\mathrm{Supp}(\gamma) = \{(3,2),(4,0),(0,2)\}$

The support is now a subset of $\mathcal{P}(\mathbf{N}^m)$.

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Example
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The support is now a subset of $\mathcal{P}(\mathbf{N}^m)$.

As before the goal is to understand $\mathrm{Supp}(\mathrm{Sol}(I))$ where I is a differential ideal of $k[[t_1,\cdots,t_m]]\{y_1,\cdots,y_n\}$.

In order to tropicalization a differential polynomial, we have to define the tropicalization of a formal serie. Let $\gamma \in k[[t_1, \cdots, t_m]]$, we define

$$\mathrm{trop}(\gamma) = \mathrm{Vert}(\mathrm{Supp}(\gamma)).$$

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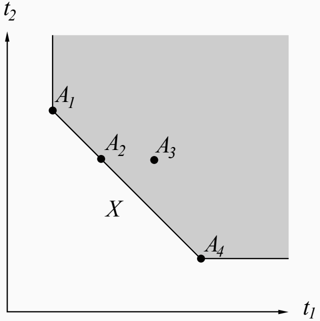
$$\mathrm{trop}(\gamma) = \mathrm{Vert}(\mathrm{Supp}(\gamma)).$$

With $\mathrm{Vert}(S)$ the vertex of the Newton polytope associated with $S \in \mathcal{P}(\mathbf{N}^m)$.

Example

Examples in $\mathbf{N}^2$ of the notion of Vert.



Newton polytope of X = Convex hull of $X + \mathbf{N}^m$

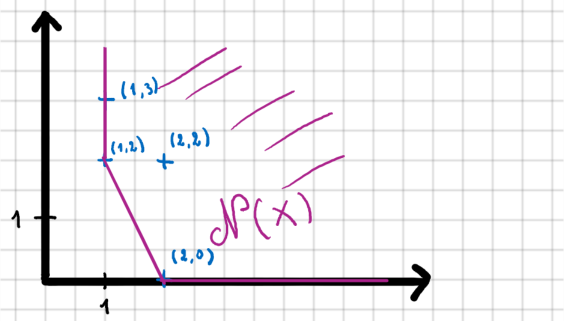
Si $X = \{A_1, A_2, A_3, A_4\}$ alors $\mathrm{Vert}(S) = \{A_1, A_4\}$.

- If $m = 1$ then $\mathrm{Vert}(S) = \min(S)$ for $S \in \mathcal{P}(\mathbf{N})$.

Example

If $\gamma = t_1t_2^2 + t_1^2 + t_1t_2^3 + t_1^2t_2^2$ then

$$\mathrm{trop}(\gamma) = \mathrm{Vert}\{(1,2) \cup (2,0) \cup (1,3) \cup (2,2)\} = \{(1,2) \cup (2,0)\}$$



Tropicalization of differentials polynomials

Let $F \in k[[t_1, \cdots, t_m]]\{y\}$. If $F = \sum_M a_M E_M$ with $a_M \in k[[t_1, \cdots, t_m]]$ and E_M a monomial in y and its derivative. Then

$$\mathrm{trop}(F) = \sum_M \mathrm{trop}(a_M) E_M.$$

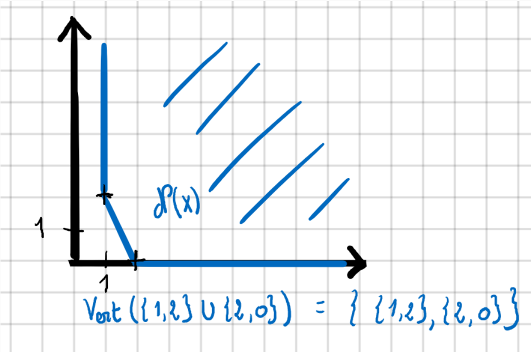
With $\mathrm{trop}(a_M) \in \mathcal{P}(\mathbf{N}^m)$.

Example

If $f = (t_1 t_2^2 + t_1^2)\frac{\partial^2 y}{\partial t_1^2}\frac{\partial y}{\partial t_2} + \frac{\partial^2 y}{\partial t_1 \partial t_2}$ then

$\mathrm{trop}(t_1 t_2^2 + t_1^2) = \mathrm{Vert}\{(1,2) \cup (2,0)\}$ et

$\mathrm{trop}(f) = \{(1,2),(2,0)\} \cdot y_{2,0} y_{0,1} + \{(0,0)\} \cdot y_{1,1}.$



Let $S \in \mathcal{P}(\mathbf{N}^m)$ et $P \in R_{m,n}$.

Goal : evaluate $\text{trop}(P)$ at S.

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Goal : evaluate $\mathrm{trop}(P)$ at S.

The operations are in $(\mathcal{P}(\mathbf{N}^m), \mathrm{Vert}, +)$.

Let $J \in \mathbf{N}^m$, we define $y_J(S) = (S - J)_+$. If $E_M = \prod_J y_J^{m_J}$, then

$$E_M(S) = \sum_J m_J \cdot y_J(S)$$

where $\sum$ is the Minkowski sum of sets.

Examples

1. If $S = \{(1,0),(2,1)\}$ then

$$y_{(1,1)}^2(S) = 2 \cdot (\{(0,-1),(1,0)\})_+ = \{(2,0)\}.$$

2. If $S = \{(0,0)\}$ then $y_{1,0}(S) = (\{(-1,0)\})_+ = \emptyset$.

Let $S \in \mathcal{P}(\mathbf{N}^m)$ et $P = \sum_M a_M E_M \in R_{m,n}$.

Evaluation

$$\mathrm{trop}(P)(S) = \mathrm{Vert}\left(\bigcup_M (\mathrm{trop}(a_M) + E_M(S))\right)$$

Example

If $S = \{(2,0),(0,2)\}$ then

$(y_{1,0} + 2y_{0,1})(S) = \mathrm{Vert}(y_{1,0}(S) \cup y_{0,1}(S)) =$

$\mathrm{Vert}((S-(1,0))_+ \cup (S-(0,1))_+) = \{(1,0),(0,1)\}$

Solutions

Let $S \in \mathcal{P}(\mathbf{N}^m)$ et $P = \sum_M a_M E_M \in R_{m,n}$. We have

$$\operatorname{trop}(P)(S) = \operatorname{Vert}\left(\bigcup_M (\operatorname{trop}(a_M) + E_M(S))\right)$$

We say that S is a solution of $\operatorname{trop}(P)$ if, for every $A \in \operatorname{trop}(P)(S)$, there exists M_1 et M_2 such that

$$A \in \operatorname{trop}(a_{M_1} E_{M_1})(S)$$

and

$$A \in \operatorname{trop}(a_{M_2} E_{M_2})(S)$$

Exemple

If $f = t_1\frac{\partial y}{\partial t_1} + t_2\frac{\partial y}{\partial t_2}$ et $S = \{(0,0),(1,1)\}$.

A formal serie of the form $\gamma = \beta + \alpha t_1 t_2$ has S as support.

Then

$$\mathrm{trop}(f) = (1,0)\cdot y_{1,0} + (0,1)\cdot y_{0,1}$$

and

$$\begin{aligned}
\mathrm{trop}(f)(S) &= \mathrm{Vert}\left\{(1,0) + (S-(1,0))_+ \bigcup (0,1) + (S-(0,1))_+\right\} \\
&= \mathrm{Vert}\left\{(1,0) + (0,1) \bigcup (0,1) + (1,0)\right\} \\
&= \mathrm{Vert}\left\{(1,1) \bigcup (1,1)\right\} \\
&= \{(1,1)\}
\end{aligned}$$

Partial fundamental theorem

Boulier, Falkensteiner, Garay-López, H., Noordman, Toghani

Let k be an uncountable field, algebraically closed and of characteristic zero. Let I be a differential ideal of $k[[t_1, \cdots, t_m]]\{y_1, \cdots, y_n\}$. Then

$$\mathrm{Supp}(\mathrm{Sol}(I)) = \mathrm{Sol}(\mathrm{trop}(I)).$$

Thanks for your attention

Fuensanta Aroca, Cristhian Garay, and Zeinab Toghani.
The Fundamental Theorem of Tropical Differential Algebraic Geometry.
Pacific J. Math., 283(2):257–270, 2016.
arXiv:1510.01000v3.

J. Denef and L. Lipshitz.
Power series solutions of algebraic differential equations.
Math. Ann., 267(2):213–238, 1984.

Sebastian Falkensteiner, Cristhian Garay-López, Mercedes Haiech, Marc Paul Noordman, Zeinab Toghani, and François Boulier.
The fundamental theorem of tropical partial differential algebraic geometry, 2020.