SQISIGN
COMPACT POST-QUANTUM SIGNATURE FROM QUATERNIONS AND ISOGENIES

LFANT seminar
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IMB, Bordeaux, France

Based on a joint work with
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SQISIGN

A post-quantum signature scheme

THE STATE OF POST-QUANTUM CRYPTOGRAPHY

Six families in Round 3 NIST post-quantum competition (finalists + alternate candidates)

Lattices	4 encryption	2 signature		
Codes	3 encryption			
Multivariate		2 signature		
Isogenies	1 encryption		compact keys	poor efficiency
Hash-based		1 signature		
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Many new isogeny-base schemes since the competition

ISOGENY-BASED SIGNATURES

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	Based on	Iterations	Sig. size	Efficiency	
[Yoo+17]	SIDH assumptions	λ	$O(\lambda^2)$	slow	
[GPS17] GPS	endomorphism computation	λ	$O(\lambda^2)$	no. implem.	weaker assumptions
[DG19] SeaSign	CSIDH assumption	λ	tradeoff	very slow	
[BKV19] CSI-FiSh	CSIDH assumption	λ	tradeoff	efficient	subexp. precomp.

λ is the security parameter

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Secret key (bytes)	Public key (bytes)	Signature (bytes)	**Security**
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New security assumption

SUPERSINGULAR ELLIPTIC CURVES

Isogenies, endomorphisms and quaternions

ELLIPTIC CURVES

Elliptic curve over $\mathbb{F}_q$: solutions (x,y) in $\mathbb{F}_q$ of

$$y^2 = x^3 + ax + b$$

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The degree* is $\deg(\varphi) = \#\ker(\varphi)$

* for separable isogenies

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Multiplication by $m \in \mathbb{Z}$ is an endomorphism

$$[m] : E \longrightarrow E : P \longmapsto P + ... + P$$

It forms a subring $\mathbb{Z} \subset \mathrm{End}(E)$

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Then, there is a $\mathbb{Z}$-basis $1, \alpha_2, \alpha_3, \alpha_4$: as a lattice,

$$\mathrm{End}(E) = \mathbb{Z} \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_4$$

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The **endomorphism algebra** is the vector space

$$B = \mathbb{Q} \oplus \mathbb{Q}\alpha_2 \oplus \mathbb{Q}\alpha_3 \oplus \mathbb{Q}\alpha_4$$

with a ring structure induced from that of End(E)

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- $\mathrm{End}(E)$ is a maximal order in $B_{p,\infty}$
- There are *many* maximal orders in $B_{p,\infty}$

DEURING CORRESPONDENCE

Supersingular curves $\boldsymbol{E}$ over $\mathbb{F}_{p^2}$	Maximal orders in $B_{p,\infty}$
(up to isomorphism)	$\mathcal{O} \simeq \mathbf{End}(\boldsymbol{E})$
	(up to equivalence)

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Degree $\mathbf{deg}(\varphi)$	Norm $\mathbf{n}(I_\varphi)$

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$$\mathrm{End}(E_0) \overset{?}{=} \mathbb{Z} \oplus \mathbb{Z}\iota \oplus \mathbb{Z}\pi \oplus \mathbb{Z}\iota\pi$$

$$\simeq \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}ij \subset B_{p,\infty}$$

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Trapdoor:

- Given $\varphi : E_0 \longrightarrow E$, easy to compute $\text{End}(E)$

The Endomorphism Ring Problem

Given a supersingular E, compute End(E)

- Hard, unless given $\varphi : E_0 \longrightarrow E$, where $\mathrm{End}(E_0)$ is known

The Endomorphism Ring Problem

Given a supersingular E, compute End(E)

⇕ heuristic

Given a supersingular E, find a non-trivial endomorphism

- Hard, unless given $\varphi : E_0 \longrightarrow E$, where $\text{End}(E_0)$ is known

AN IDENTIFICATION PROTOCOL

Proving knowledge of an endomorphism ring

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Idea first exploited in GPS Signatures in 2017

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- Prove knowledge of $\mathrm{End}(E)$ by solving such instances?

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$$E_0 \xrightarrow{\text{key } \tau} E_A$$

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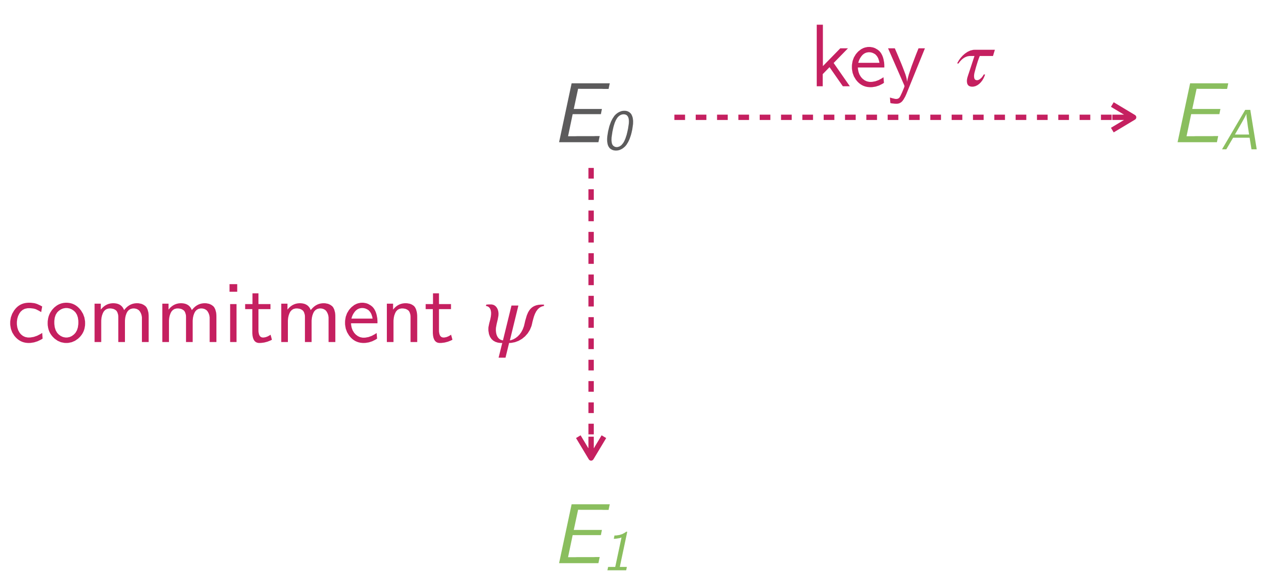

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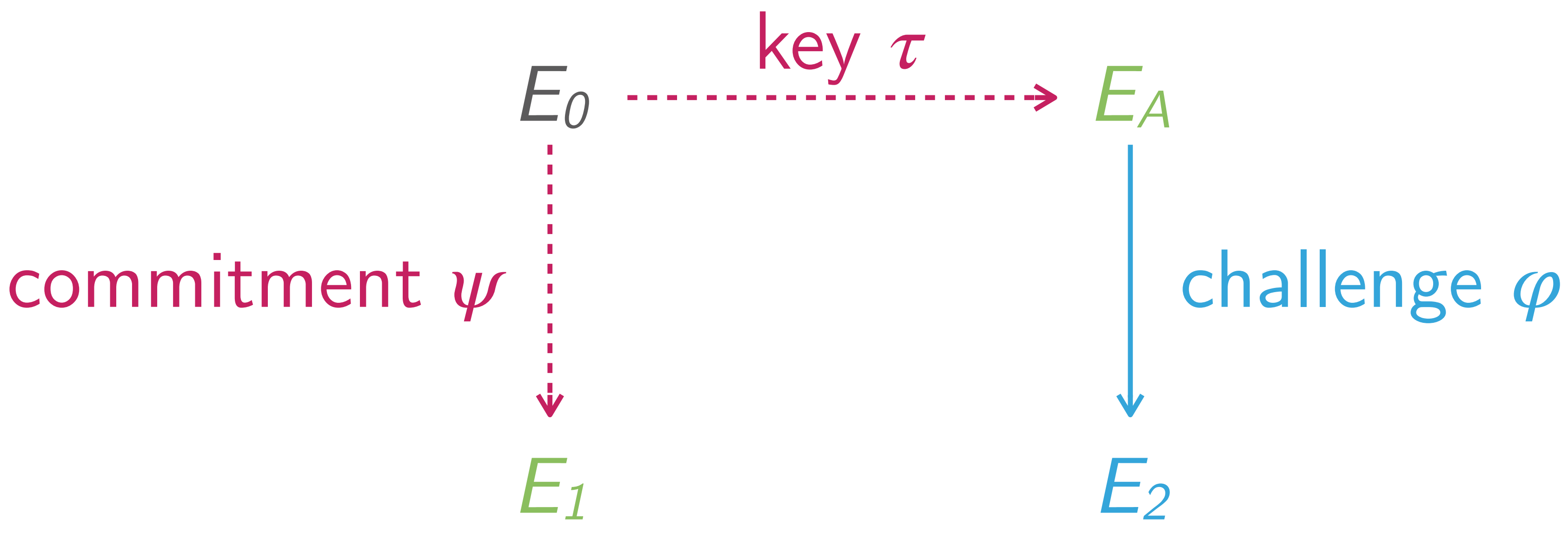

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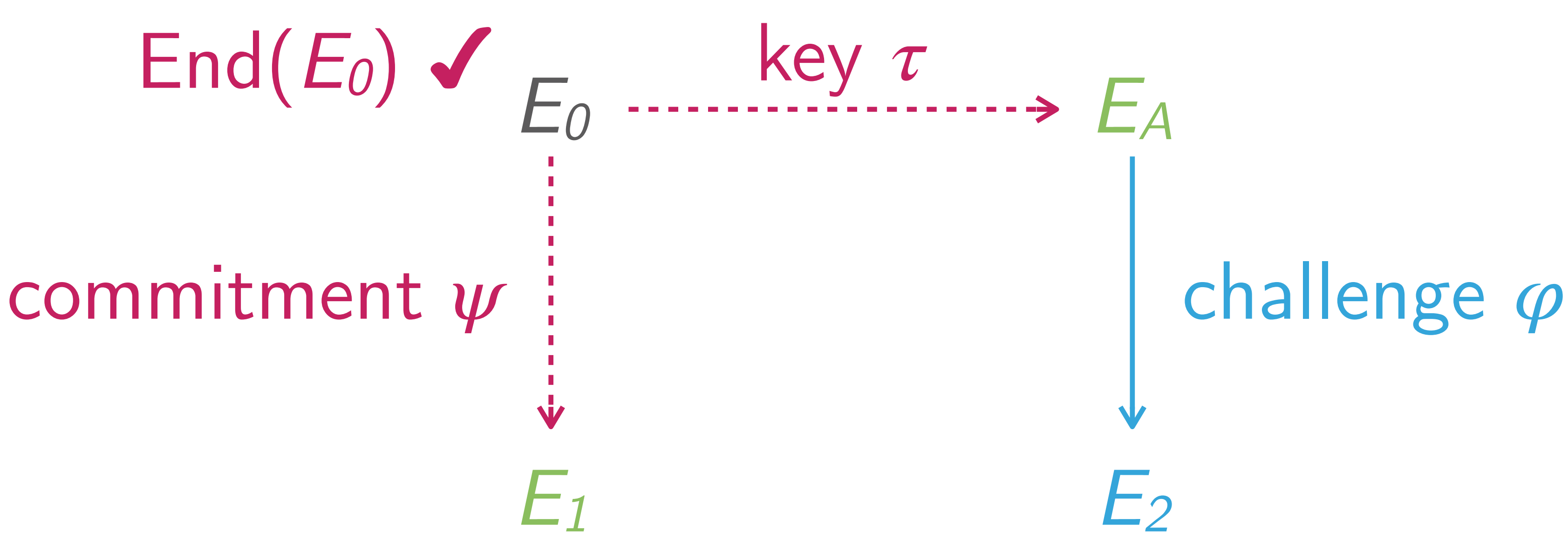

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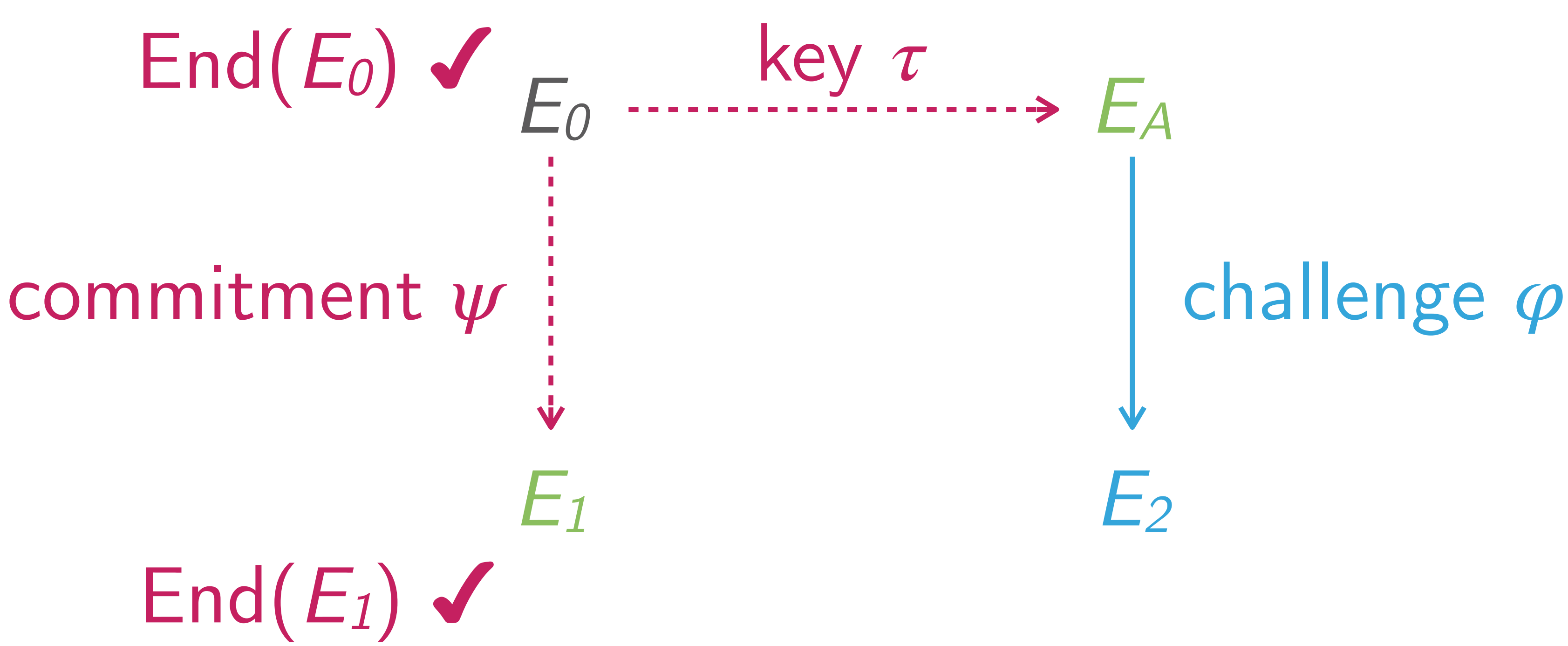

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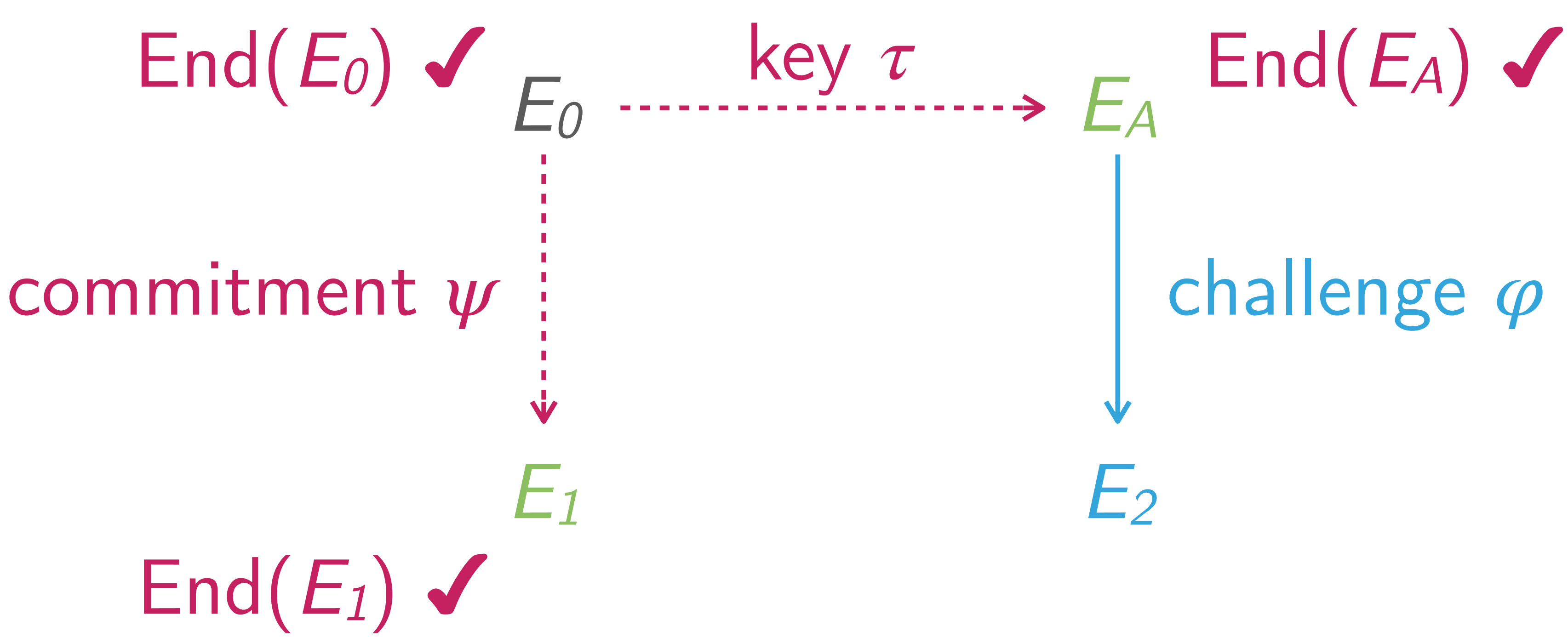

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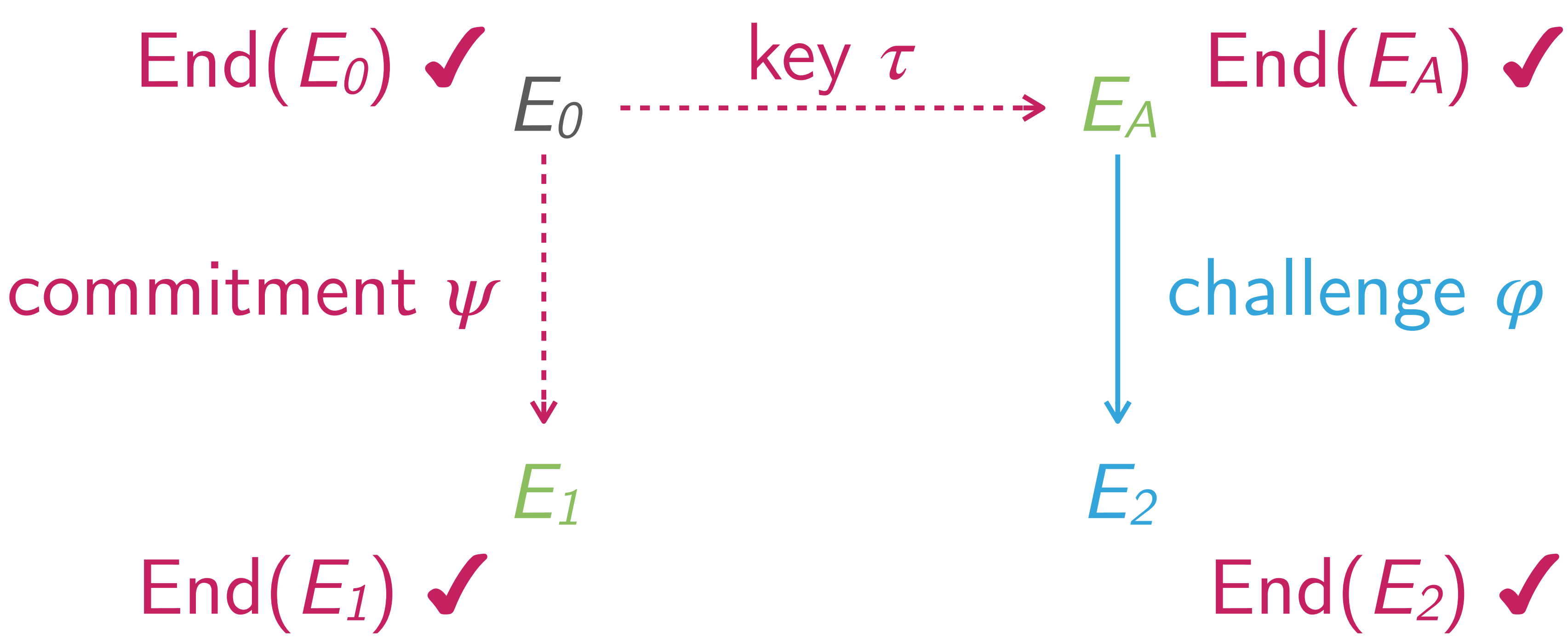

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- Can compute (secretly) $\mathrm{End}(E_A)$
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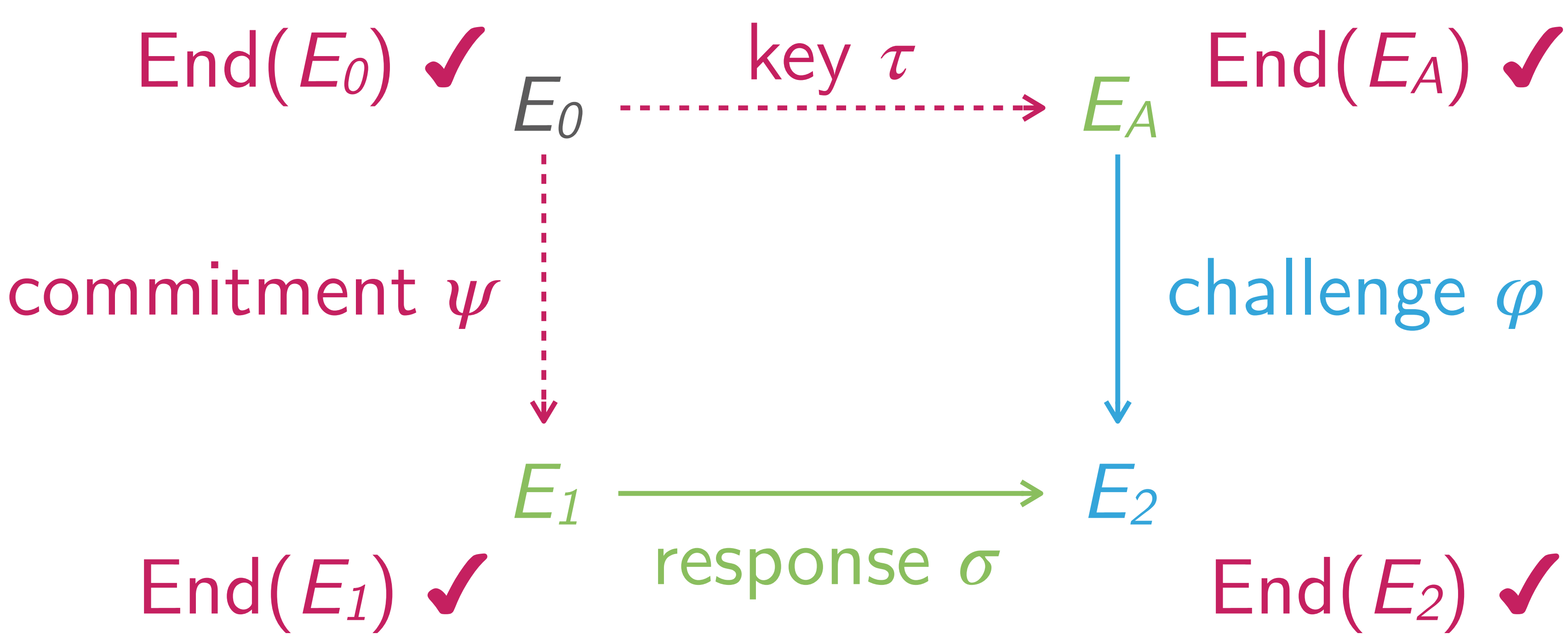

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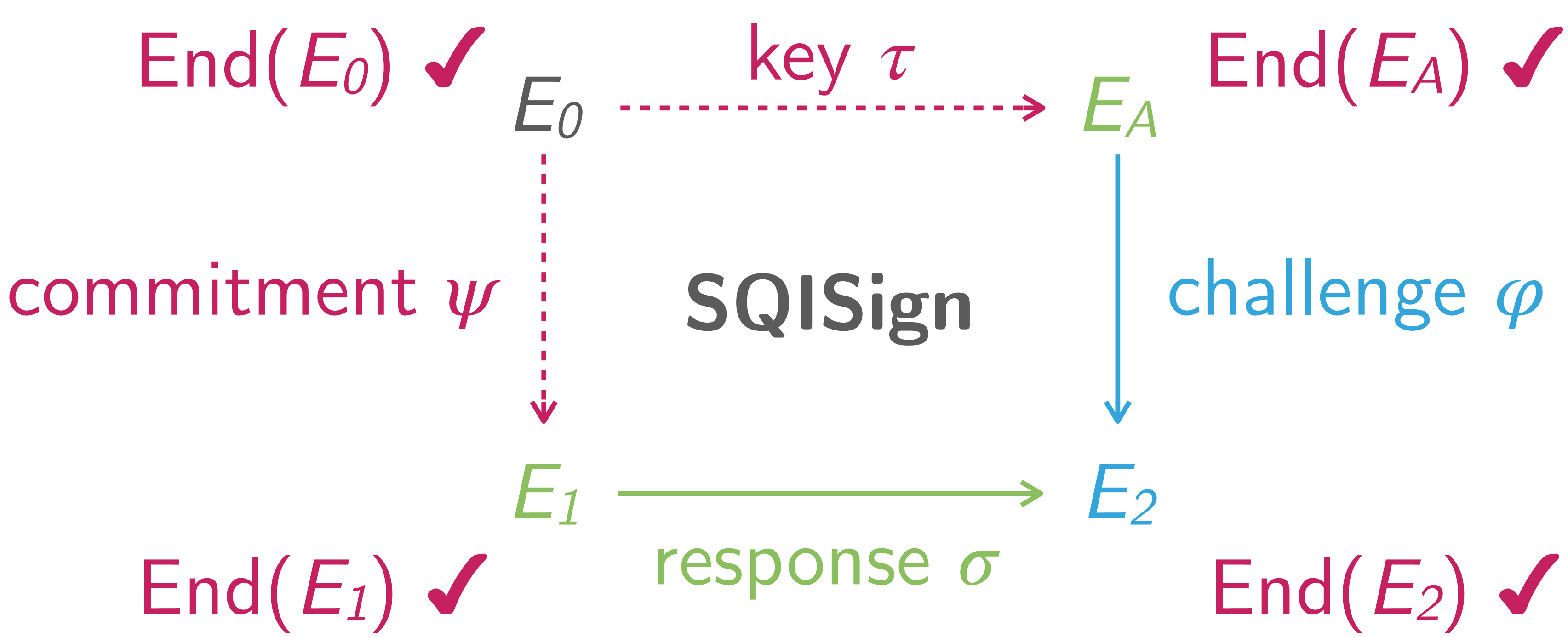
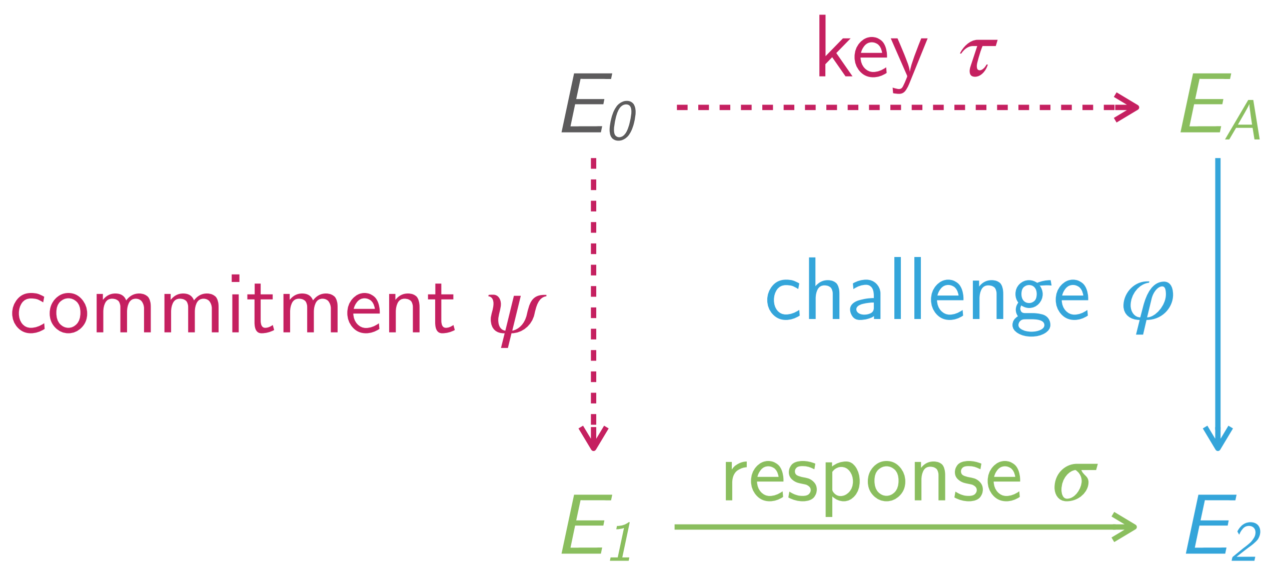


SPECIAL SOUNDNESS

Special soundness: responding to two distinct challenges (for same commitment) allows to recover the secret



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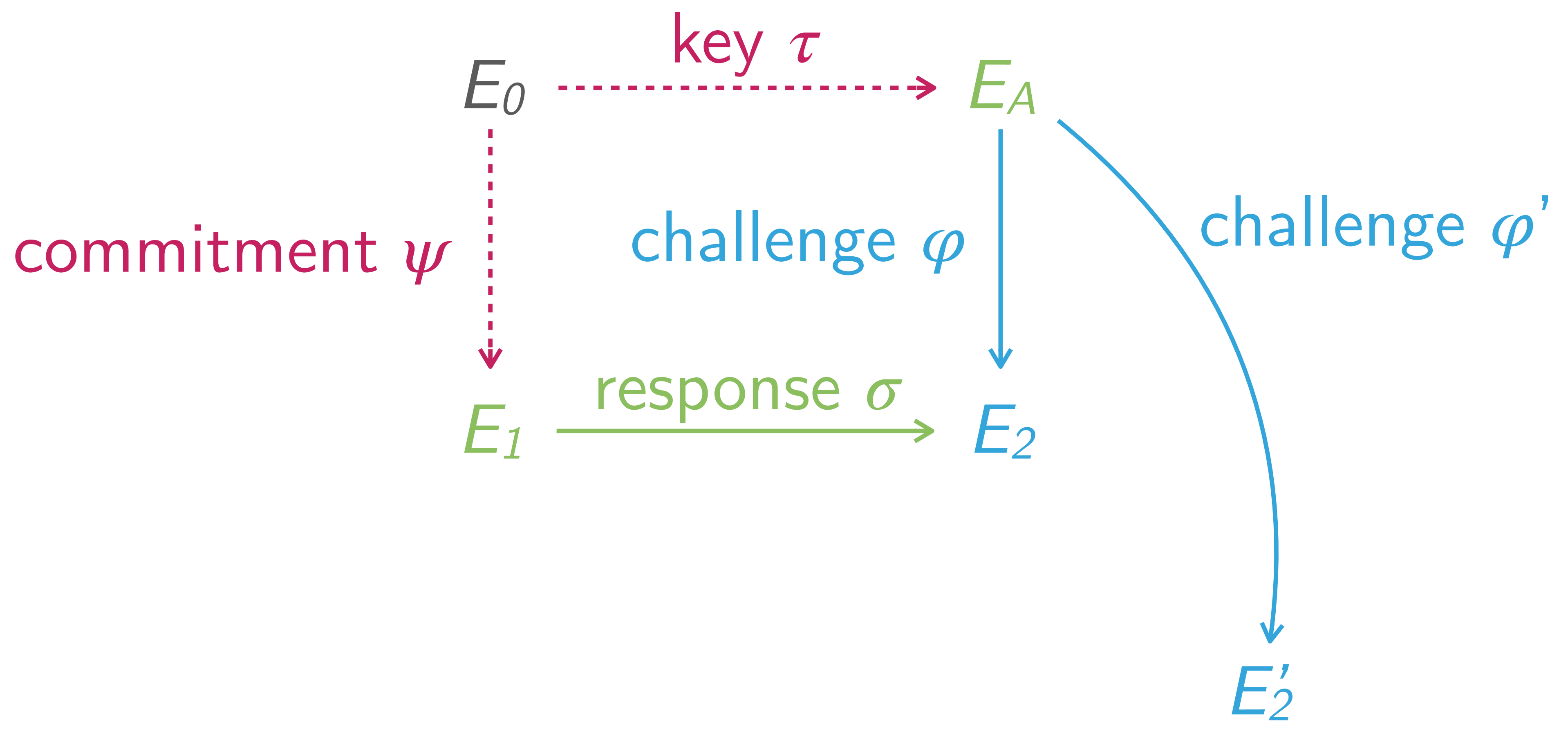

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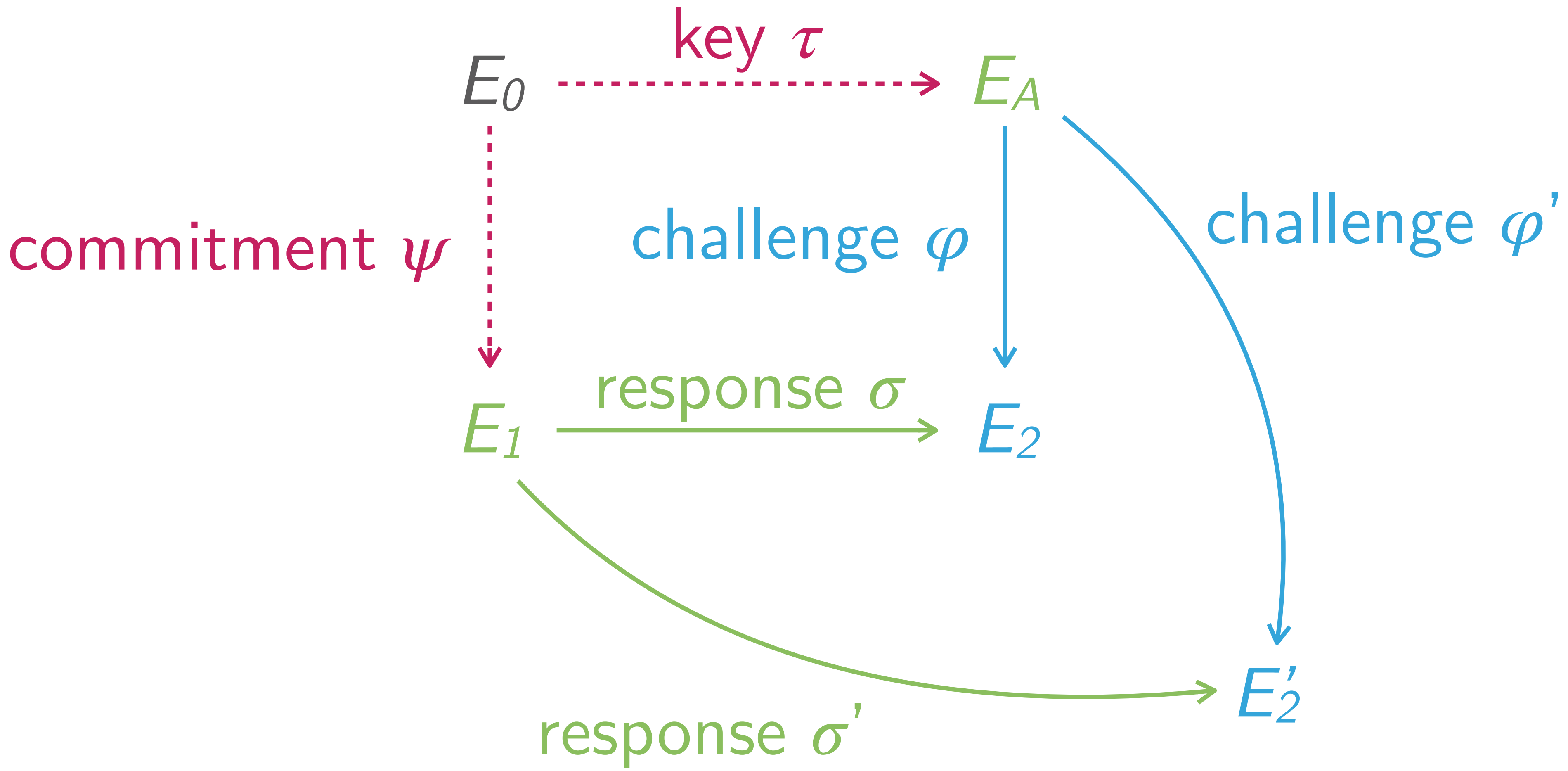

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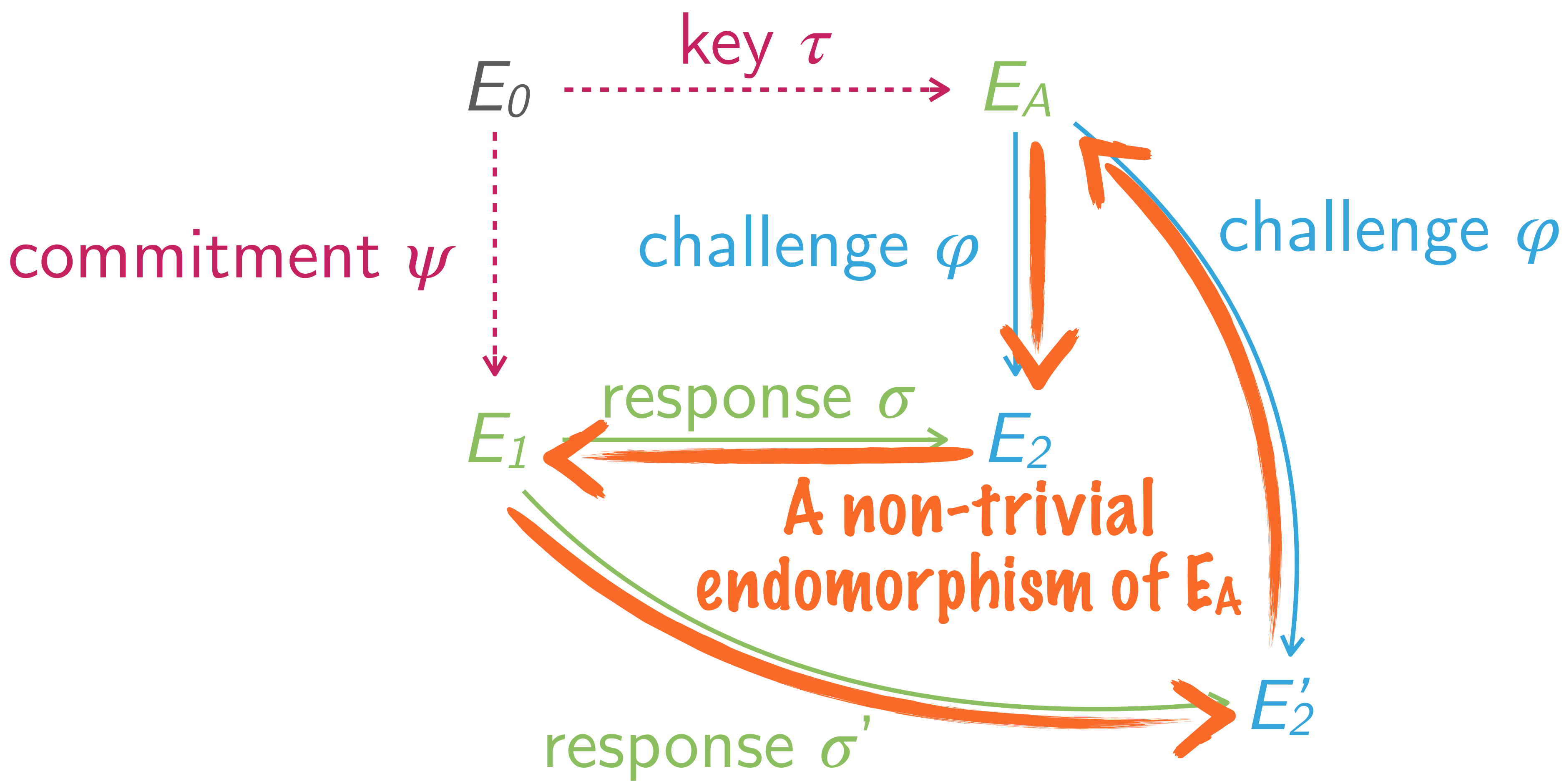

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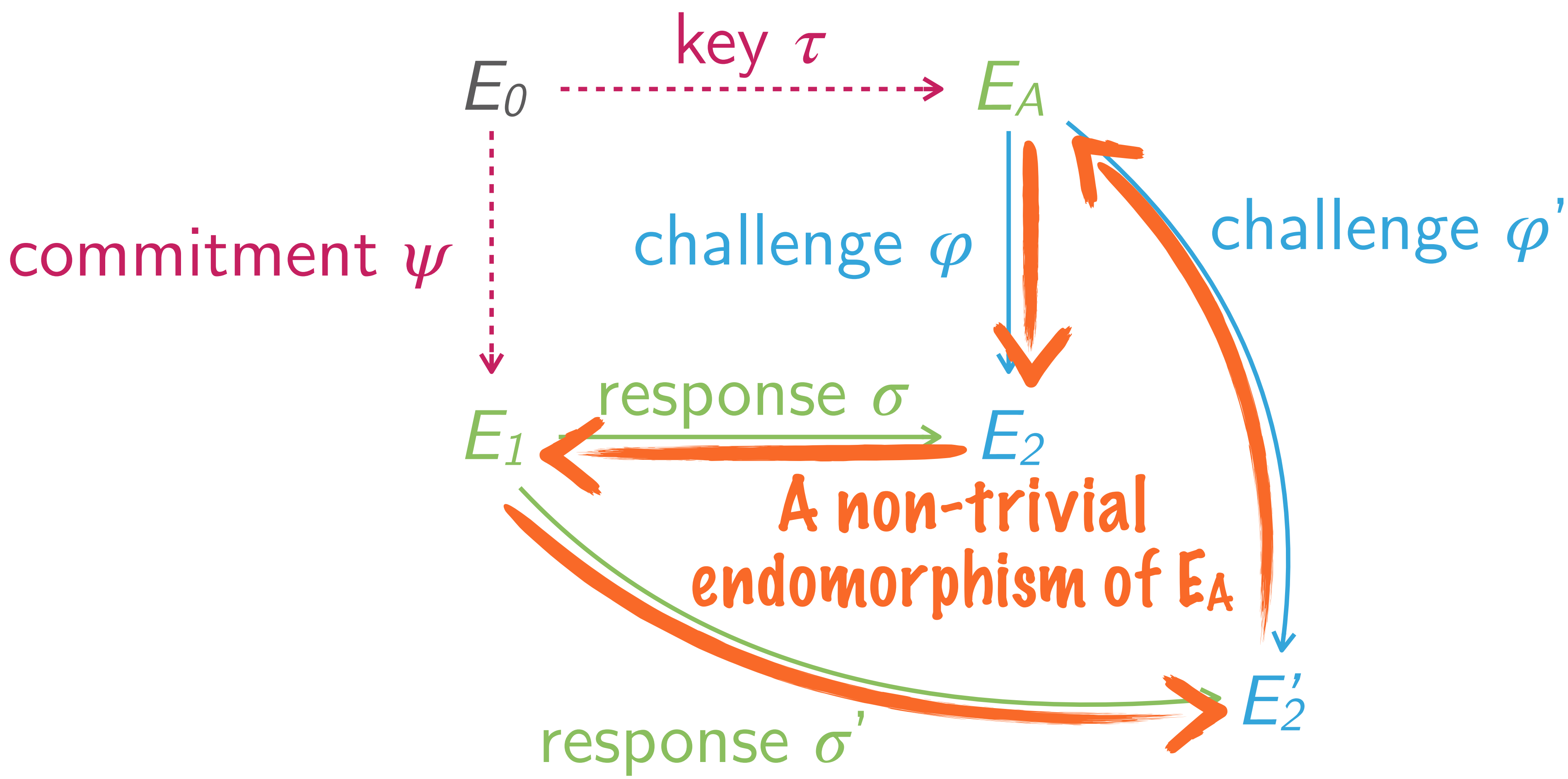

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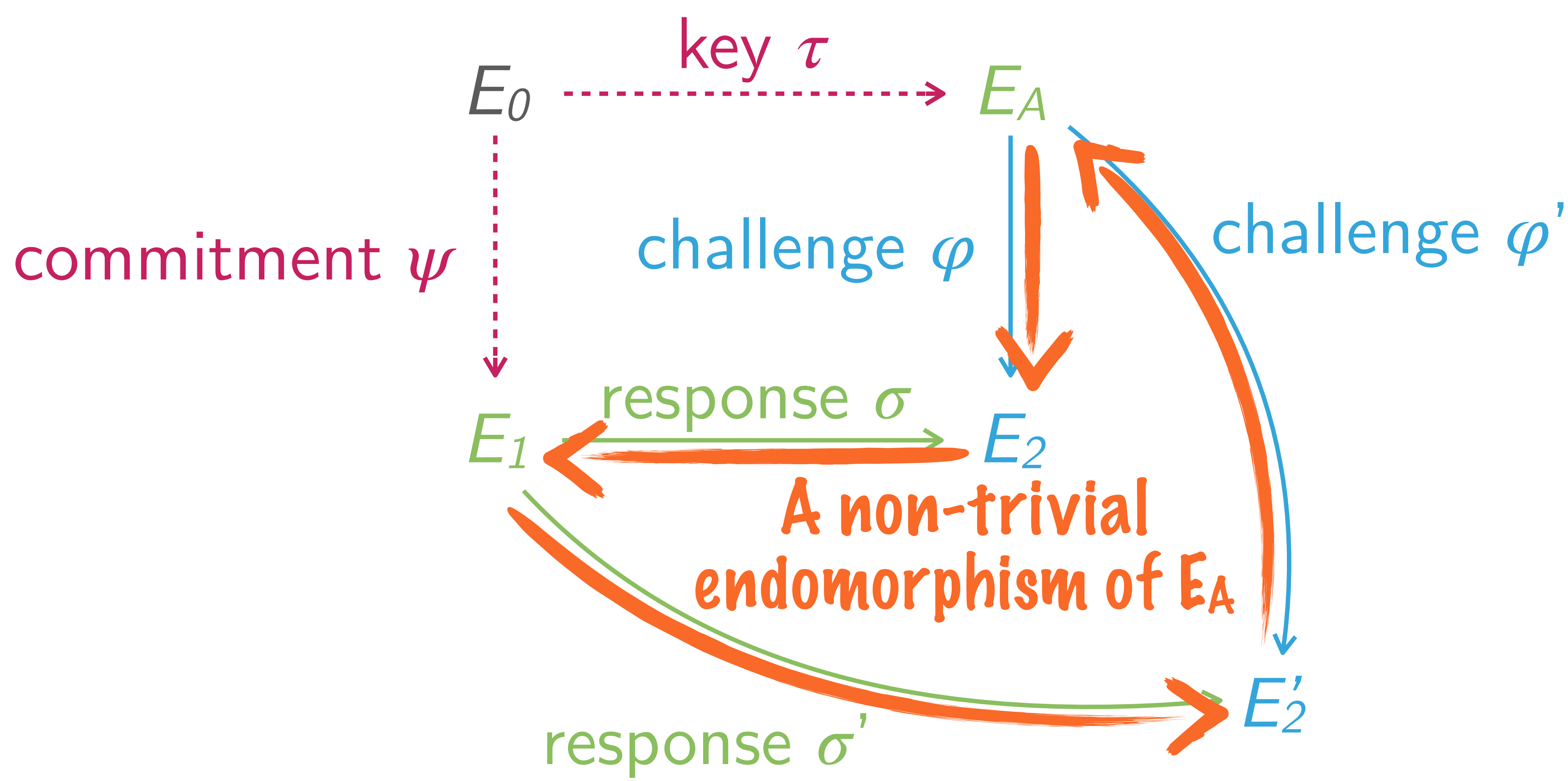

Breaking soundness $\Rightarrow$ solving "Non-trivial endomorphism" $\Rightarrow$ solving "Endo ring problem"

heuristic

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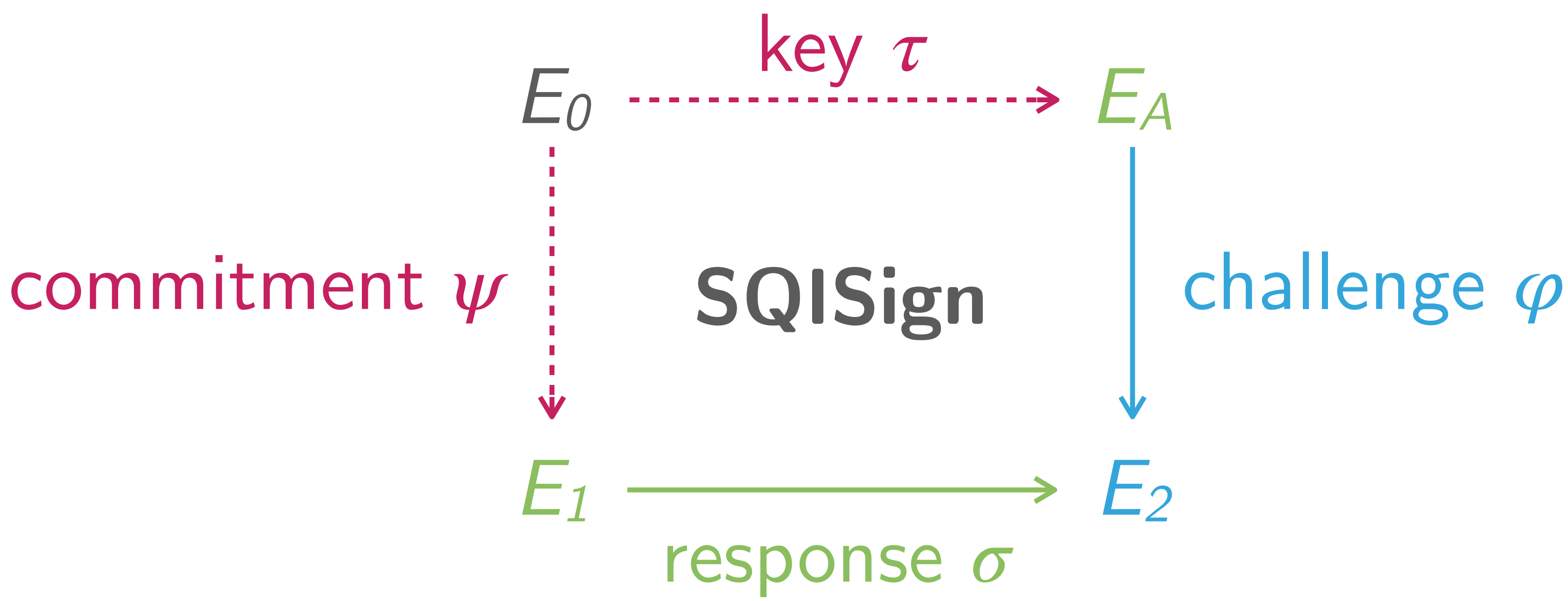

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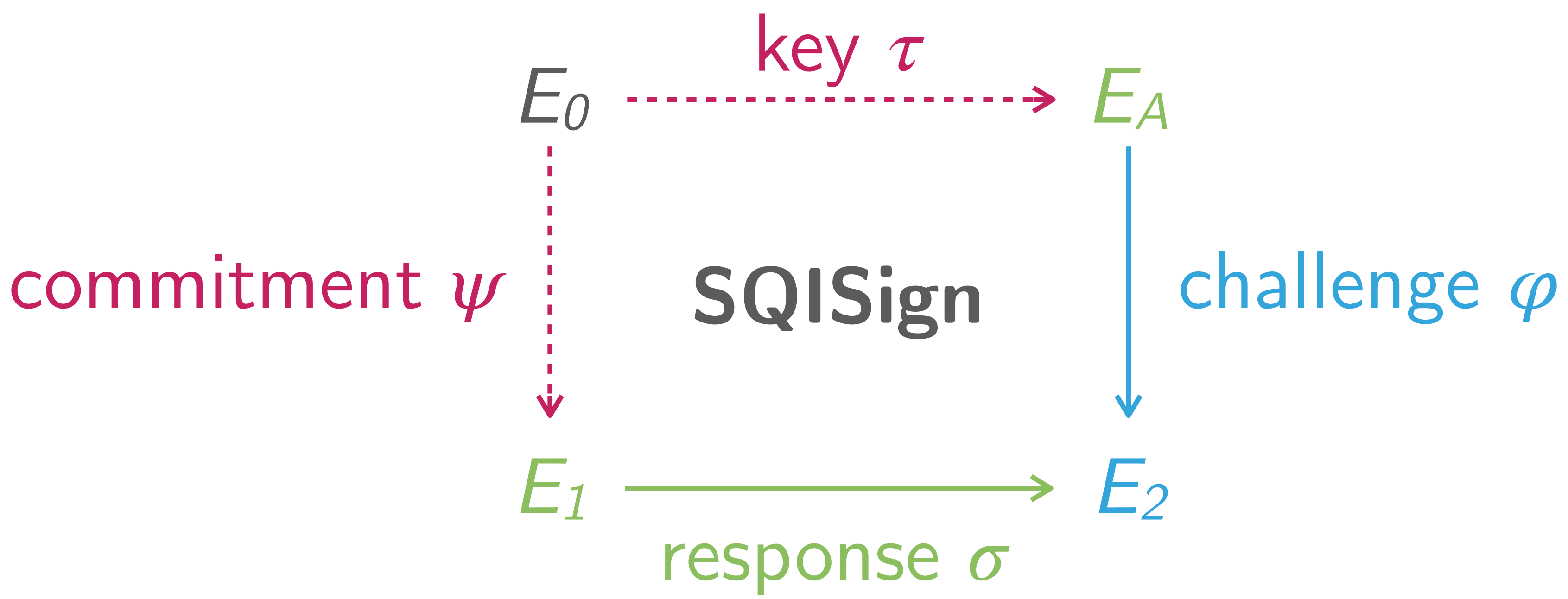

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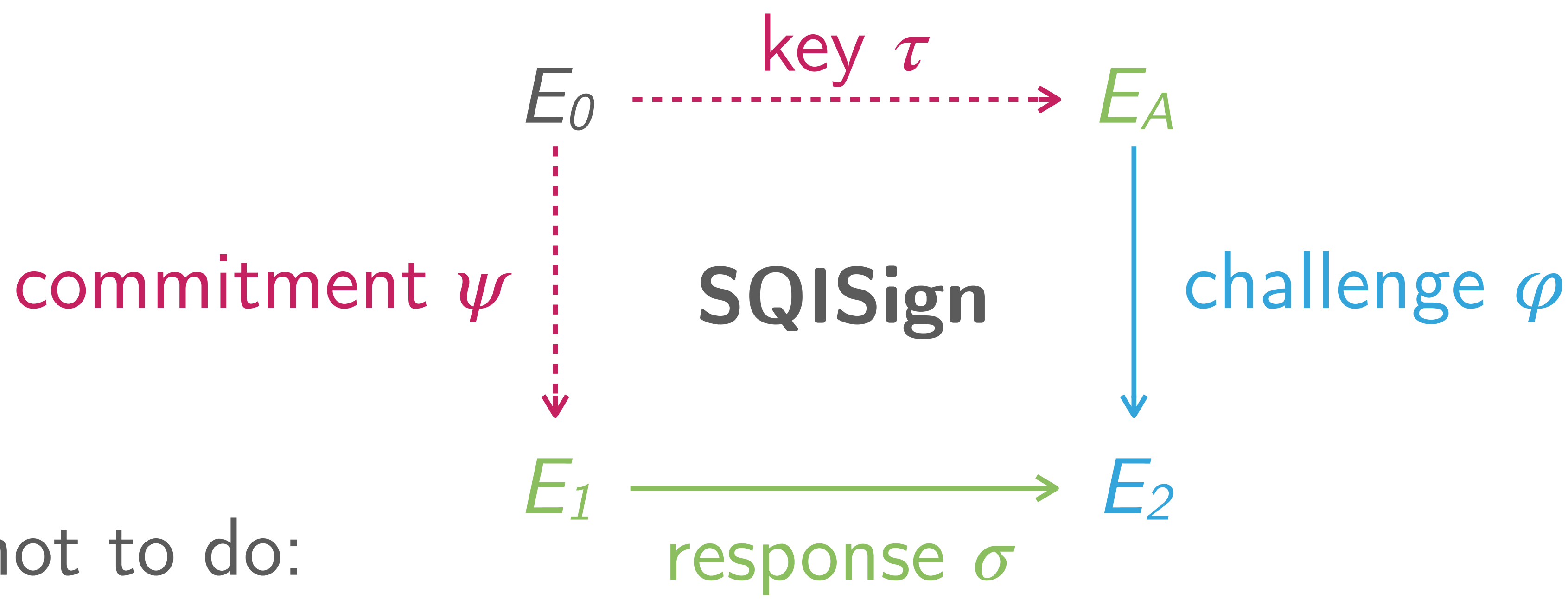

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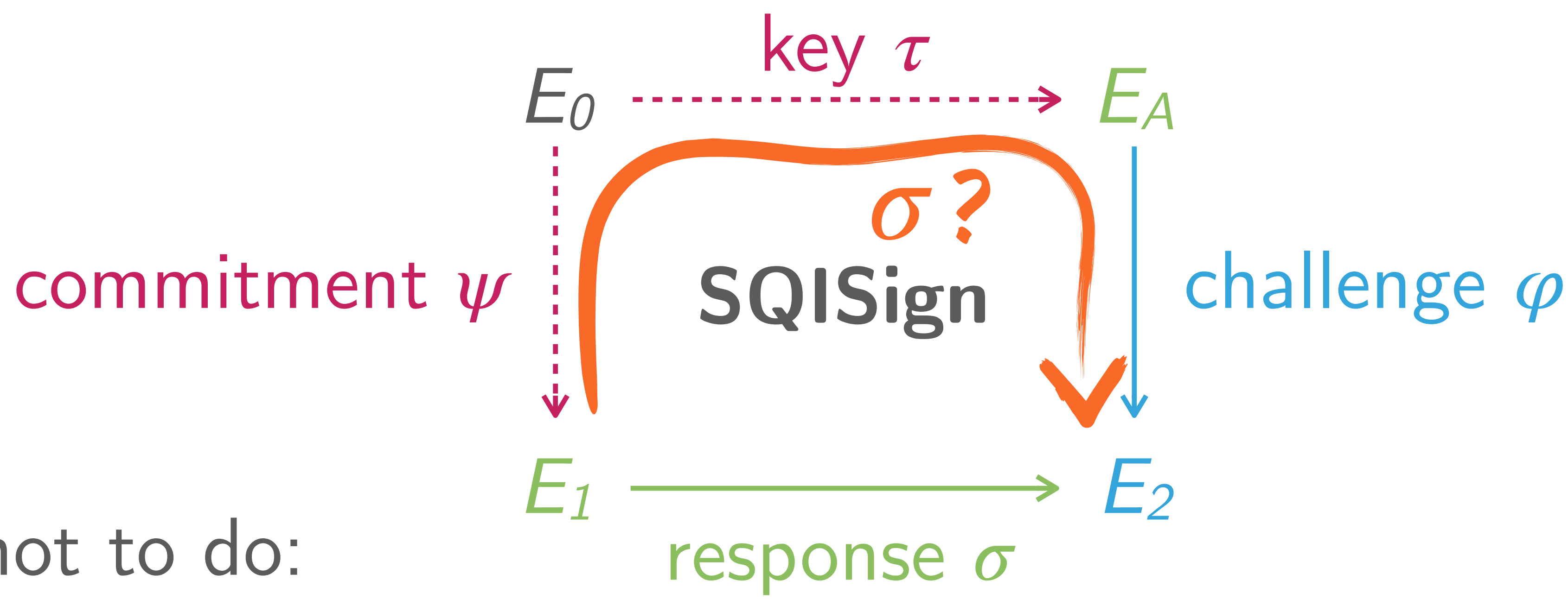

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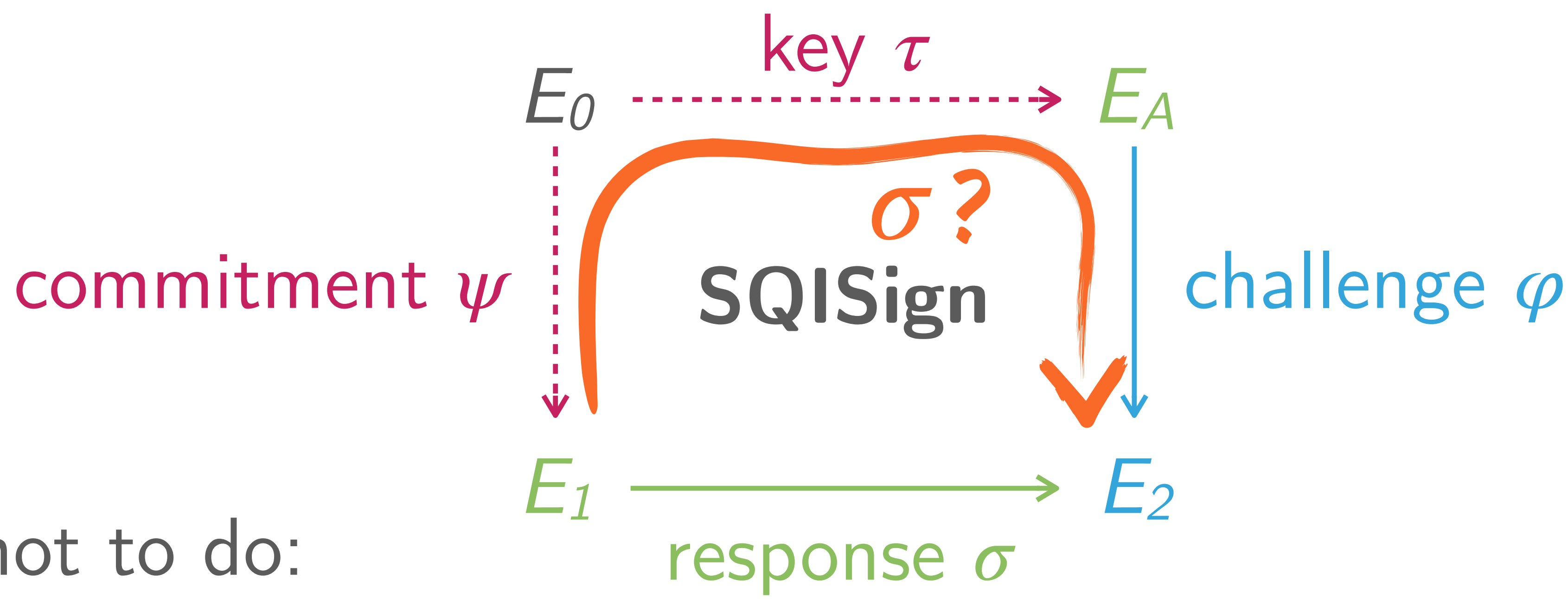

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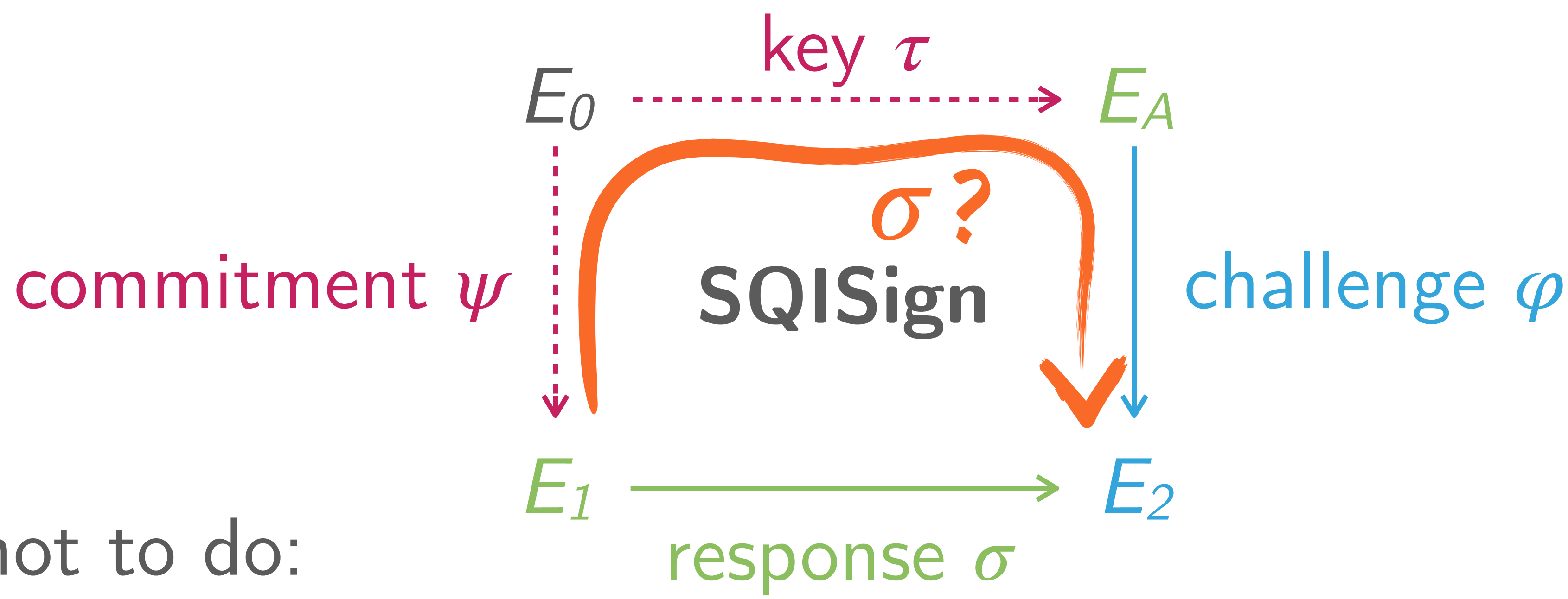

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Main technical difficulty of SQISign

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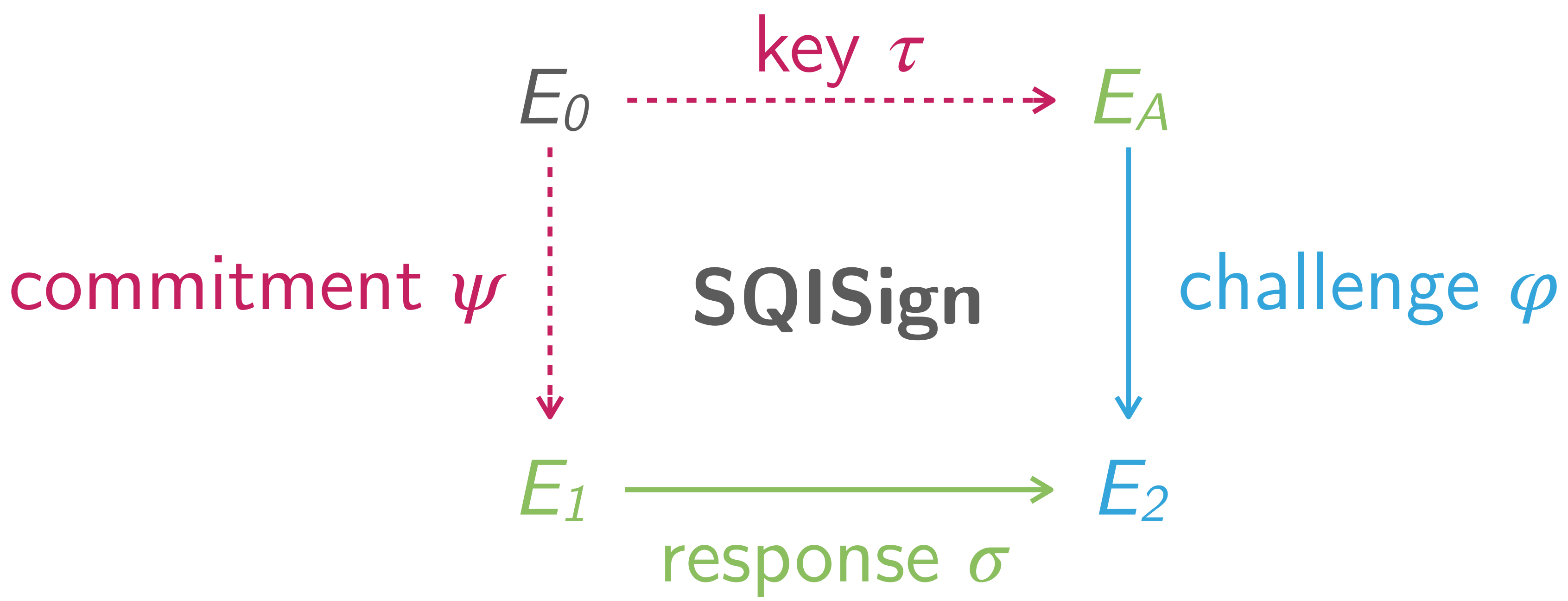
SQISIGN IN PRACTICE

How to make it fast

VERIFICATION



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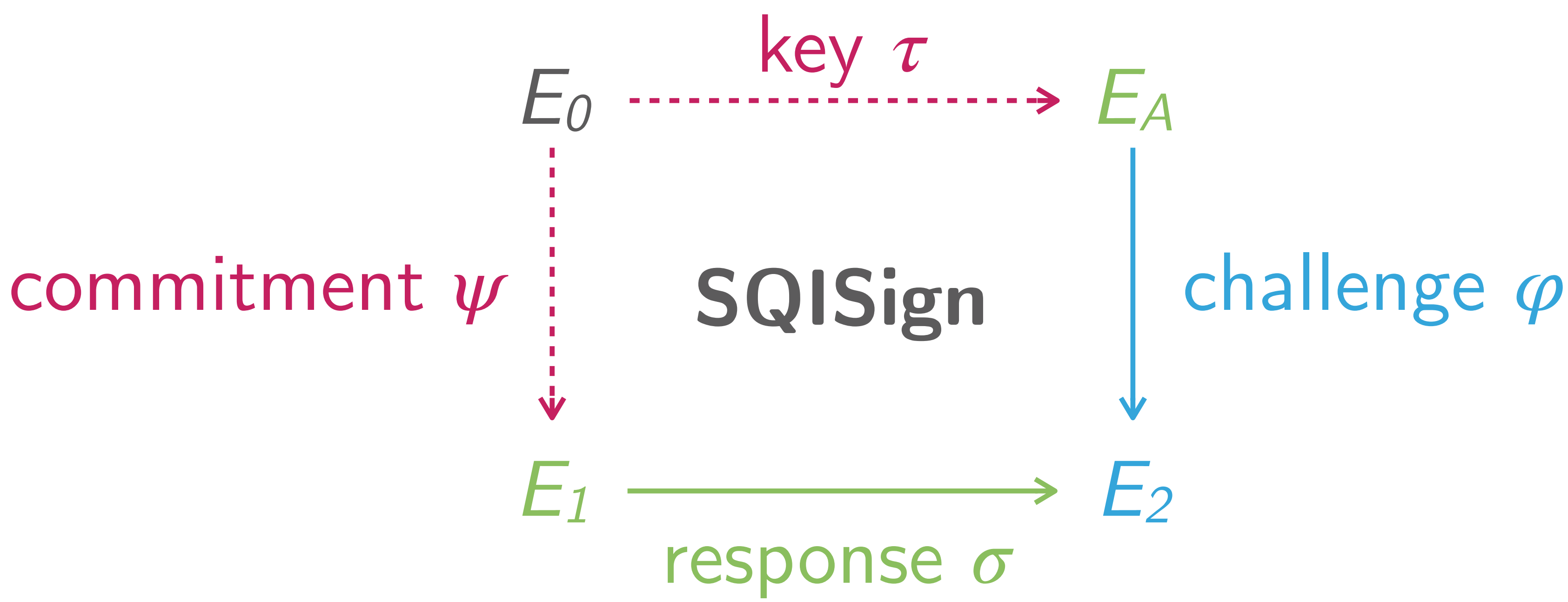

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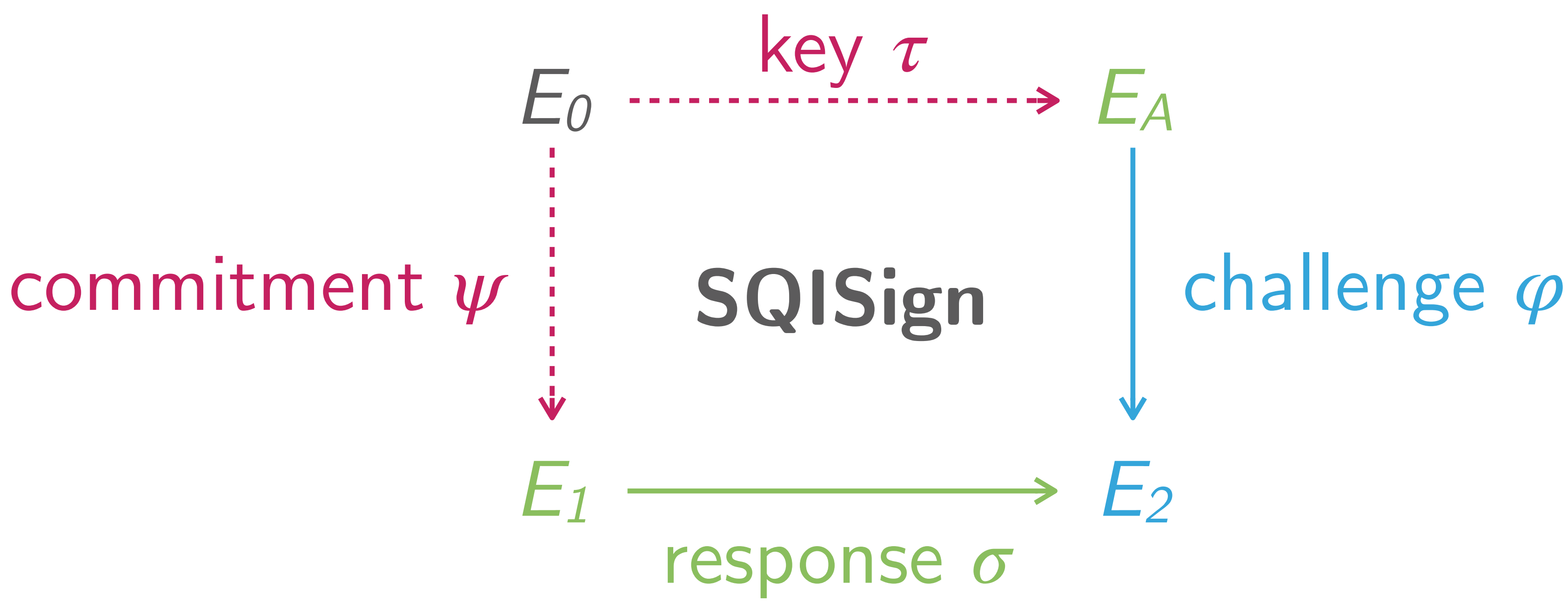

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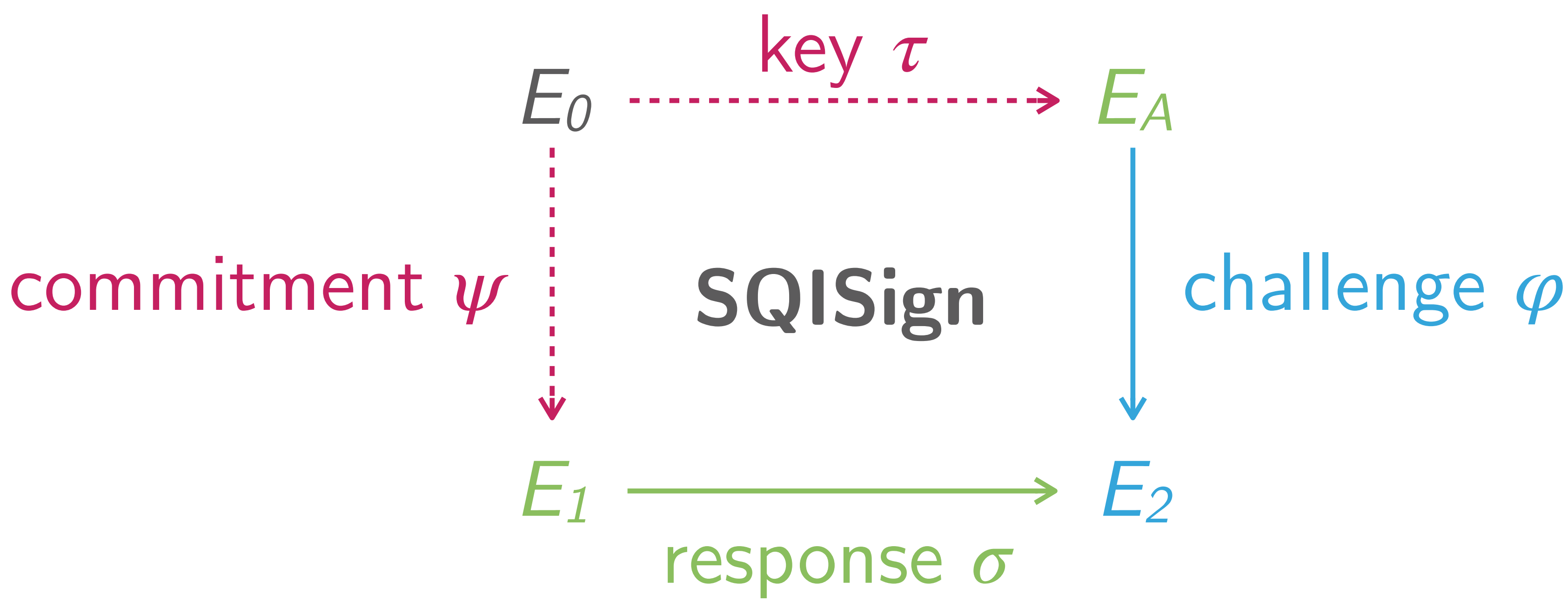

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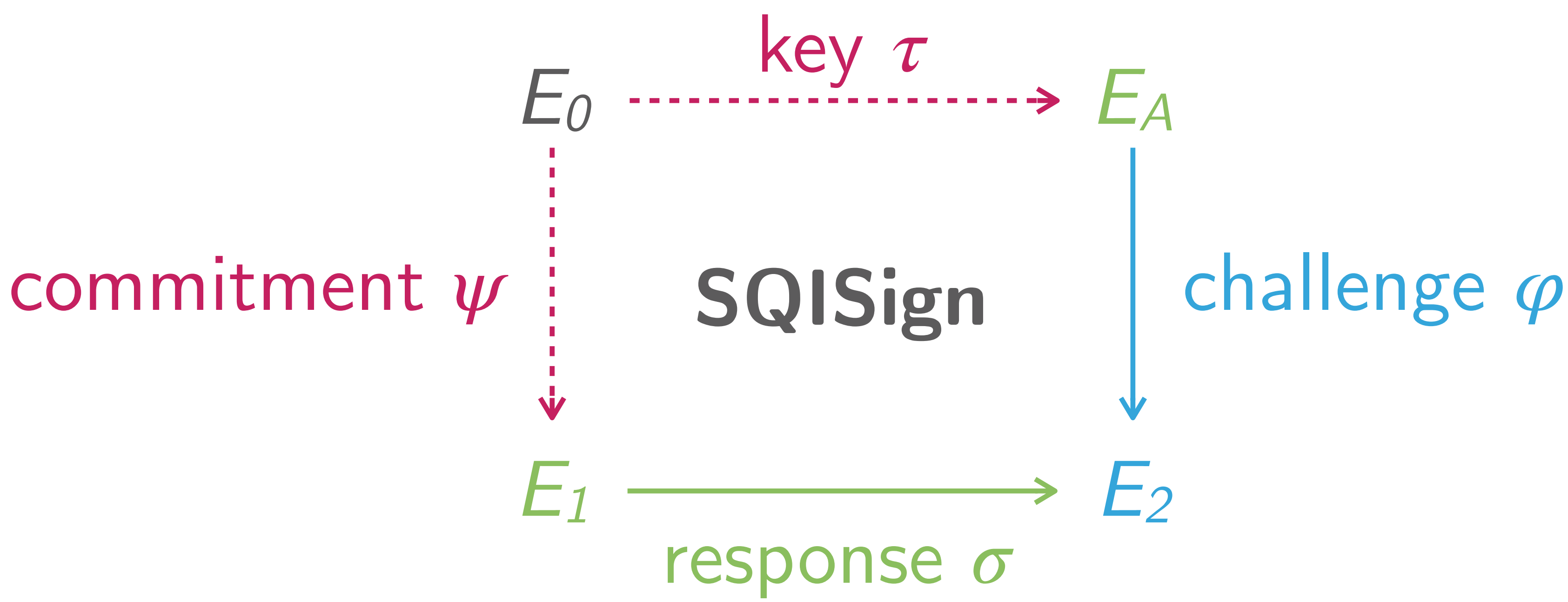

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Efficient verification! (choosing $\deg(\sigma)$ and $\deg(\varphi)$ smooth)

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Solution: new algorithmic tools, and a careful choice of the base prime p, so extensions are not necessary

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Use Chinese Remainder Theorem to enlarge search space

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We found a prime p such that $e = 33$, and

$$T = 3^{53} \cdot 43 \cdot 103^2 \cdot 109 \cdot 199 \cdot 227 \cdot 419 \cdot 491 \cdot 569 \cdot 631 \cdot 677 \cdot 857 \cdot 859 \cdot 883 \cdot 1019 \cdot 1171 \cdot 1879 \cdot 2713 \cdot 4283 \cdot 5^{21} \cdot 7^2 \cdot 11 \cdot 31 \cdot 83 \cdot 107 \cdot 137 \cdot 751 \cdot 827 \cdot 3691 \cdot 4019 \cdot 6983$$

SQISIGN
COMPACT POST-QUANTUM SIGNATURE FROM QUATERNIONS AND ISOGENIES

LFANT seminar
November 2021
IMB, Bordeaux, France

Based on a joint work with
Luca De Feo, David Kohel,
Antonin Leroux and
Christophe Petit

cnrs Inria université de BORDEAUX

Benjamin Wesolowski