Concentration in order types of random point sets

arXiv:2003.08456 (prel. version in SoCG 2020)

Xavier Goaoc — Université de Lorraine

Emo Welzl — ETH Zürich

LFANT seminar, December 14th, 2021

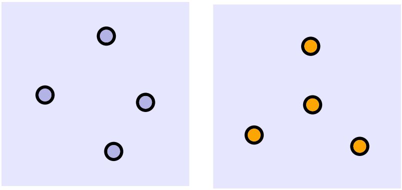
The pitch:

- $\triangleright$ a (labeled) order type is a combinatorial object induced by a finite set of points in $\mathbb{R}^2$.
- $\triangleright$ I'd like to know to **generate randomly** such an object without **excessive bias**.
- $\triangleright$ Here, I'll discuss why sampling random point sets can be **inefficient**.

Order types
& labeled order types

"What is the simplest geometric object about which one can say something interesting? A single point did not quite do it, but a finite set of points—in general position in the plane, say—did: there was an essential difference between four points that formed the vertices of a quadrilateral and three points with a fourth inside their triangle."

Jacob E. Goodman



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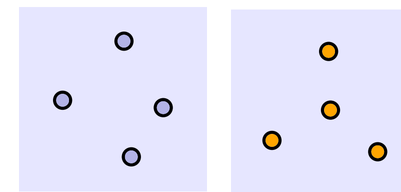

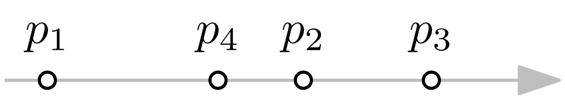

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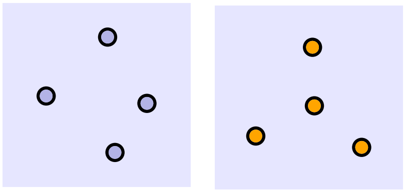

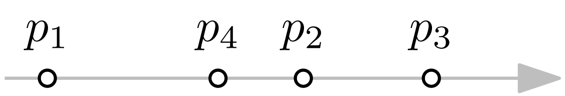

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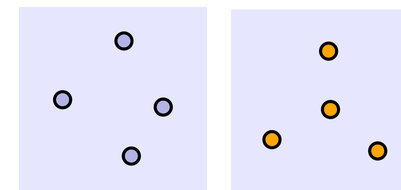

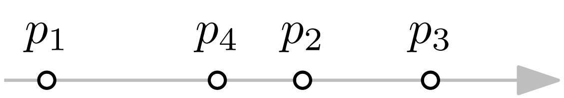

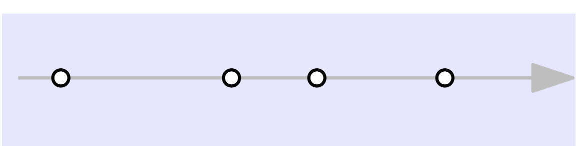

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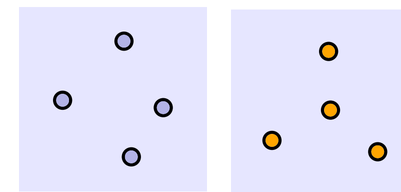

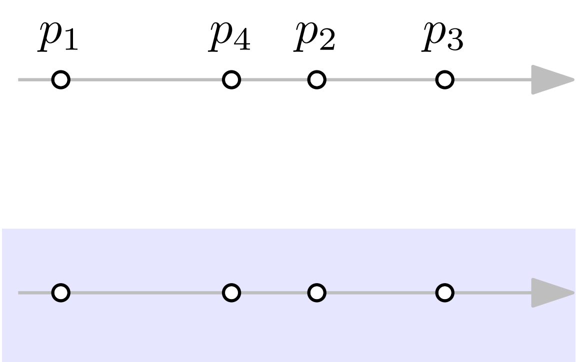

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As of dimension 2, the structure gets **richer** and **more intricate**.

Reduction of the **infinite** diversity of n-point sets in $\mathbb{R}^2$

to **finitely** many combinatorial types based on **orientation predicates**.

[Goodman-Pollack'83]

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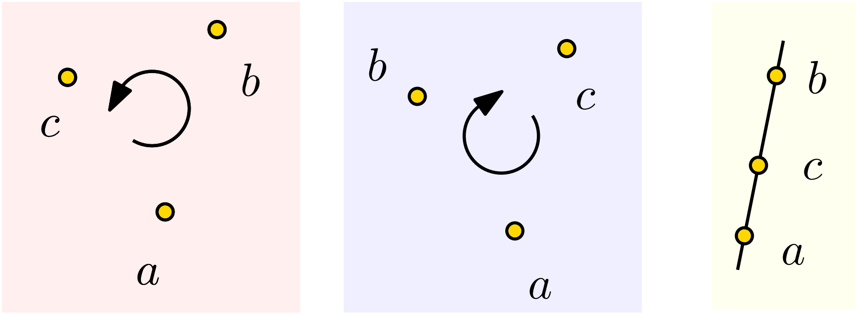

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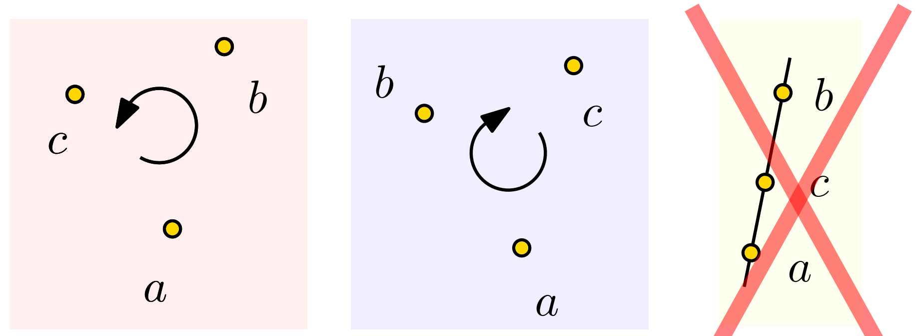

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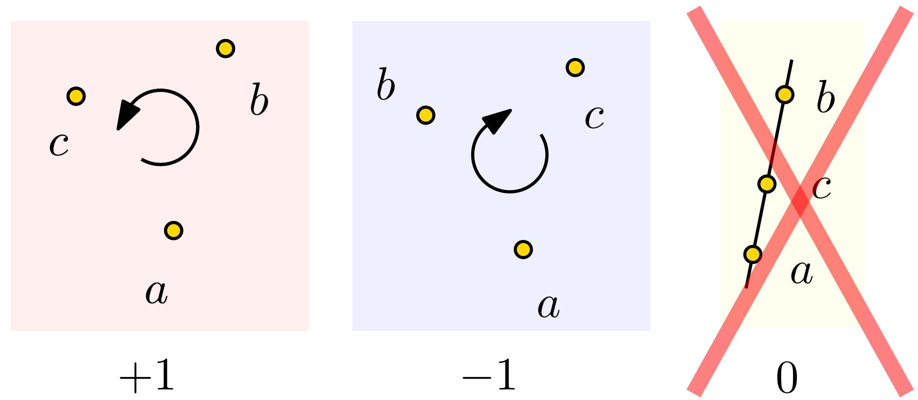

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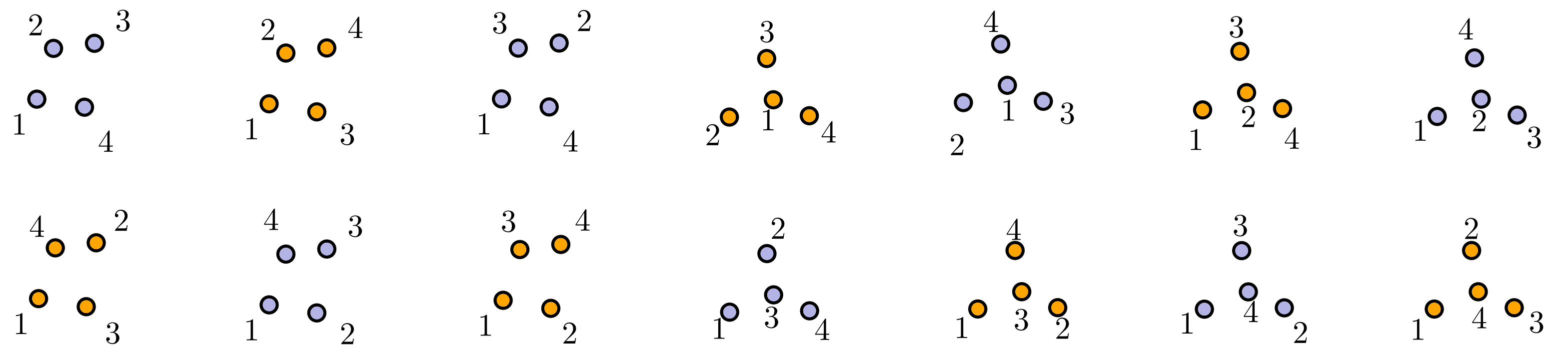
Two point **sequences** $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ such that

$$\forall 1 \leq i,j,k \leq n, \quad \chi(p_i, p_j, p_k) = \chi(q_i, q_j, q_k)$$

have the same **labeled order type**.

Equivalence relation, class $\simeq$ map: triples of indices $\to \{-1, 1\}$.

Example: representatives of the labeled order types of size 4.



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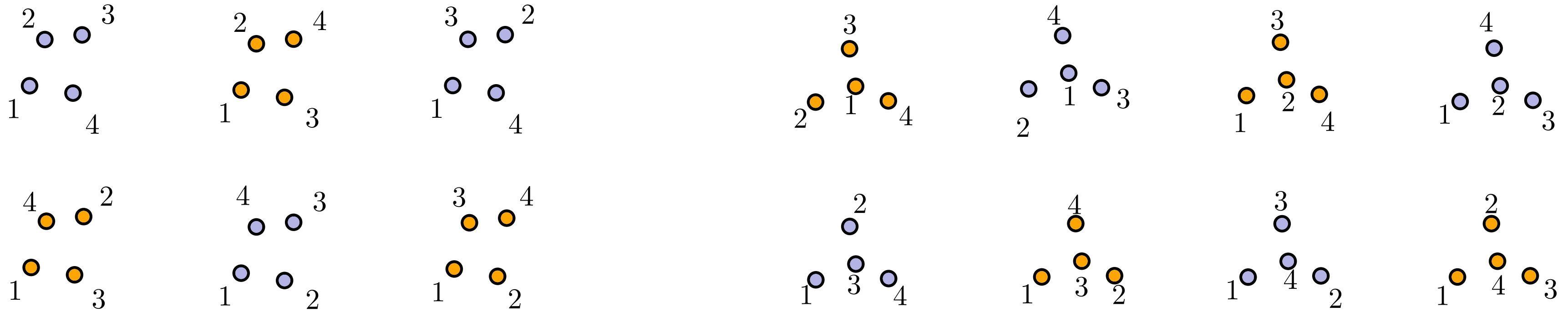

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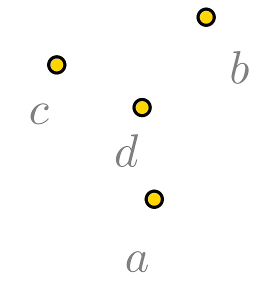

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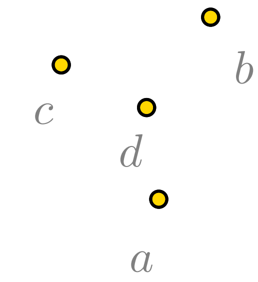

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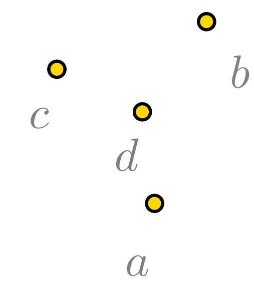

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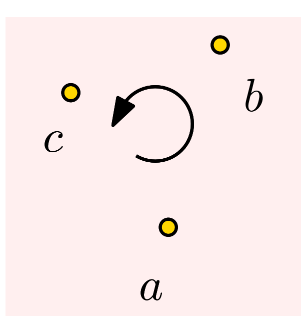

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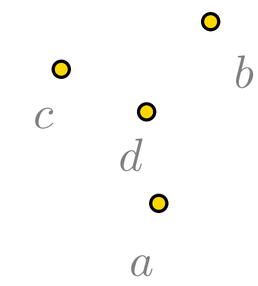

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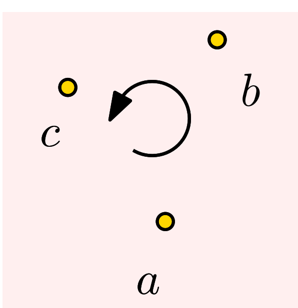

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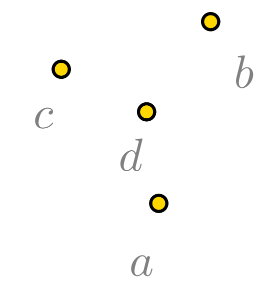

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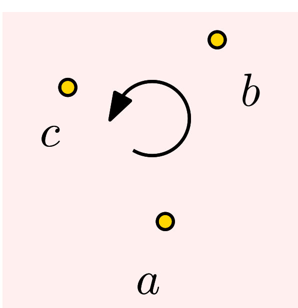

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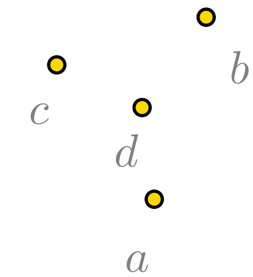

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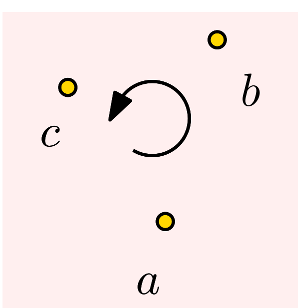

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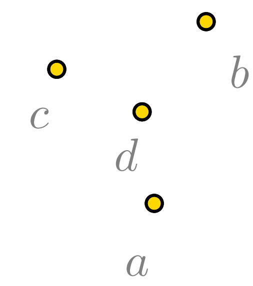

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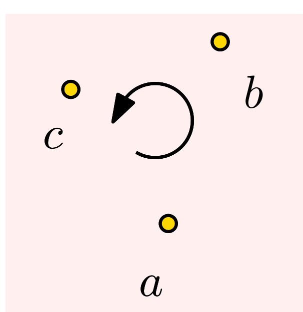

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$$\chi(a, b, c) = \text{sign of the determinant of } \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{pmatrix}$$

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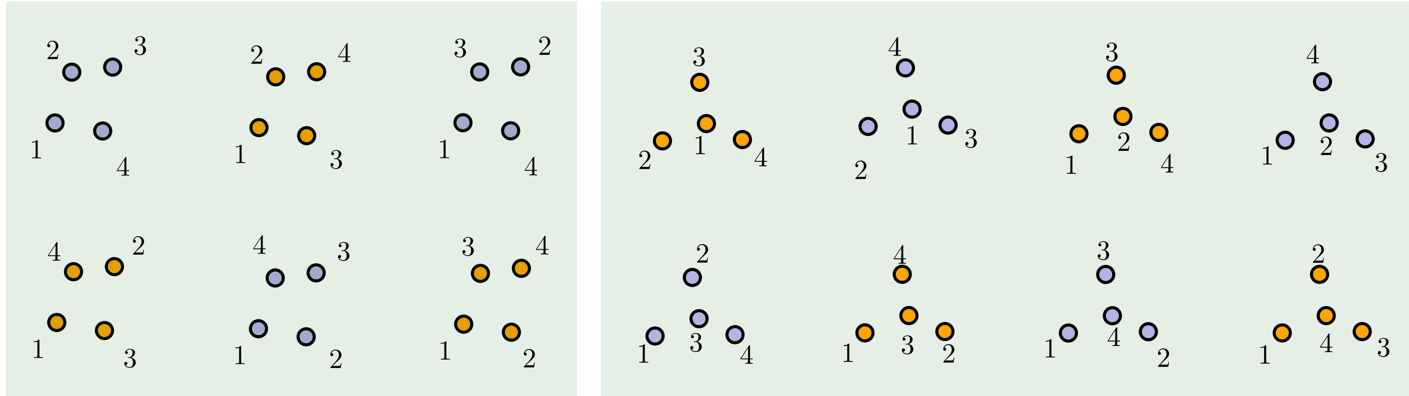
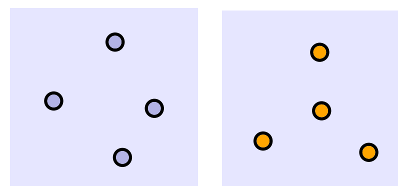


Discrete geometers

often drop the labeling.



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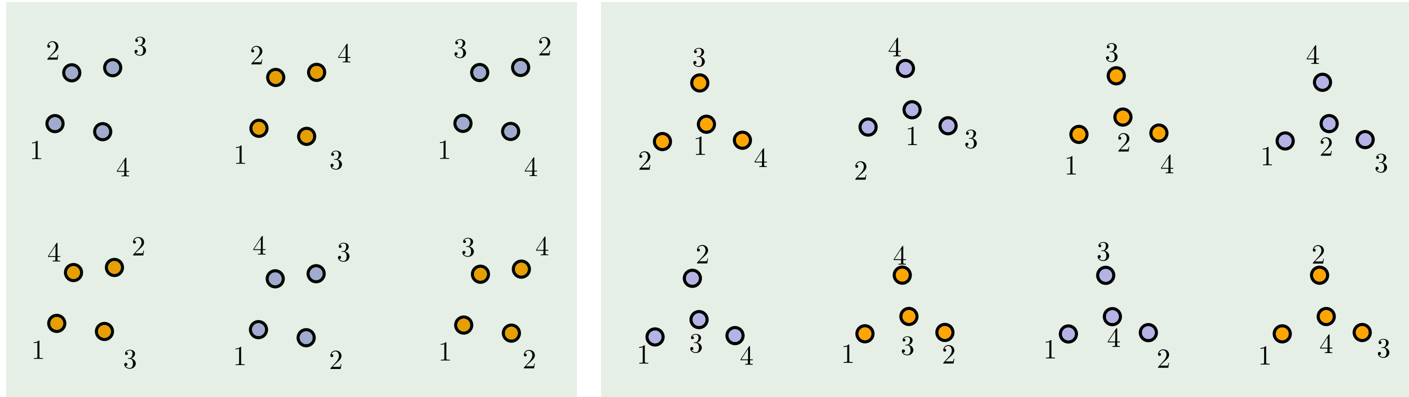

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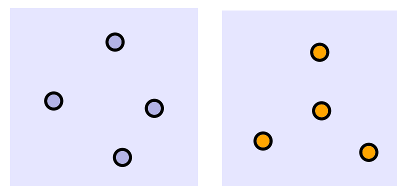


Two point **sets** P and Q have **the same order type** if there exists an **orientation-preserving** bijection between them.

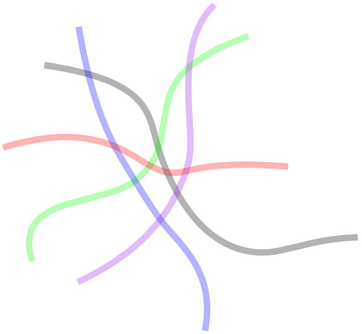
Equivalence relation, order type $\stackrel{\text{def}}{=}$ an equivalence class.

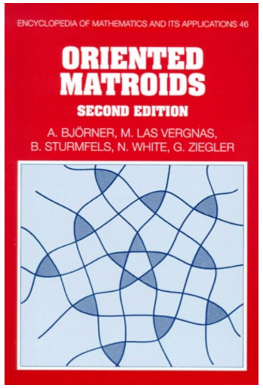
Common properties for convex hull, segments intersections, . . .

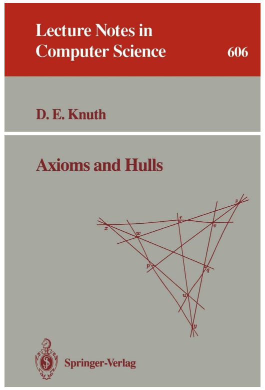
Can be defined **abstractly** in **topological affine planes**.

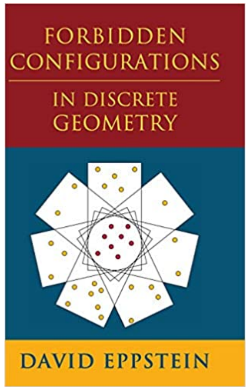
Labeled abstract order types = **acyclic uniform oriented matroids**.

[Salzmann'67][Folkman-Lawrence'78]









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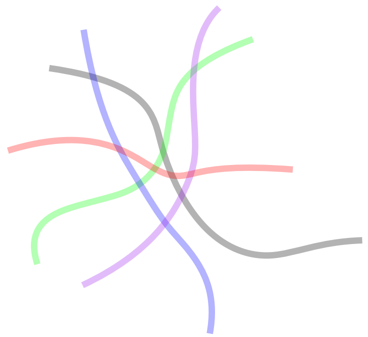


Labeled abstract order types $=$ **acyclic uniform oriented matroids**.

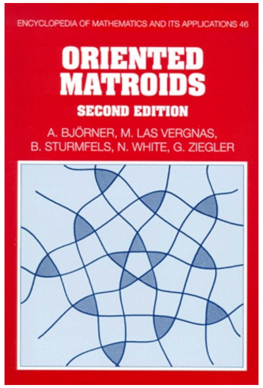
[Salzmann'67][Folkman-Lawrence'78]

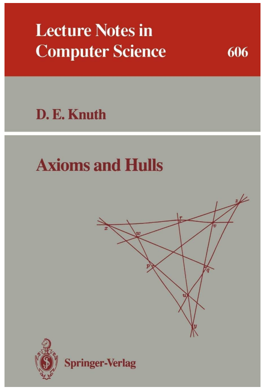
Computationally, abstract OT are simple

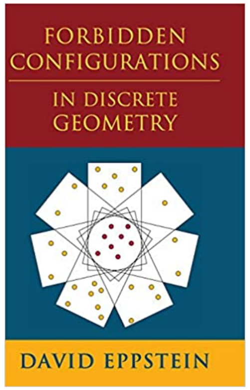
characterized by a few axioms on ≤ 5 points.

but realizable OT are complicated.

Universality [Mnëv'86], $\exists\mathbb{R}$-completeness [Shor'92]







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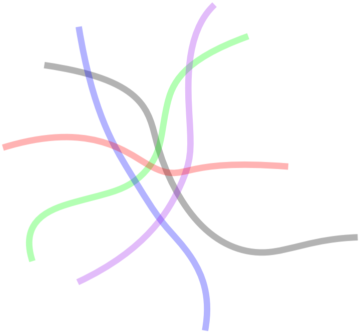

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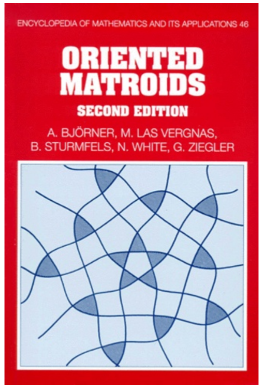

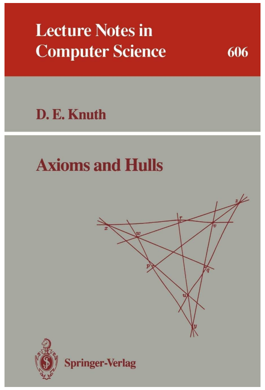

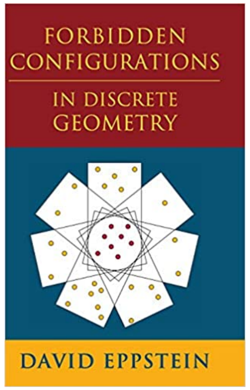

Realizable OT were **enumerated** up to size 11.

3	4	5	6	7	8	9	10	11
1	2	3	16	135	3 315	158 817	14 309 547	2 334 512 907

(mirror images are identified)

[Aichholzer-Aurenhammer-Krasser'02]

but even the **number** of (abstract) n-point order types is unknown.

Sampling order types

Given n, how should we generate a **random** (labeled) order type of size n?

Not mere intellectual curiosity...

- $\triangleright$ Tool to test problems/conjectures expressible by orientation predicates.

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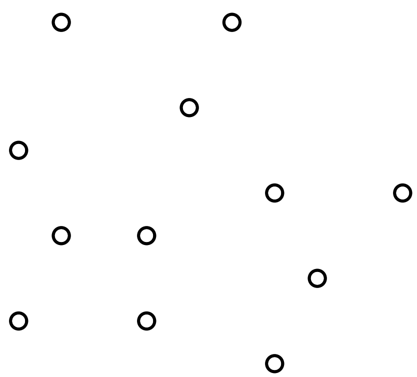

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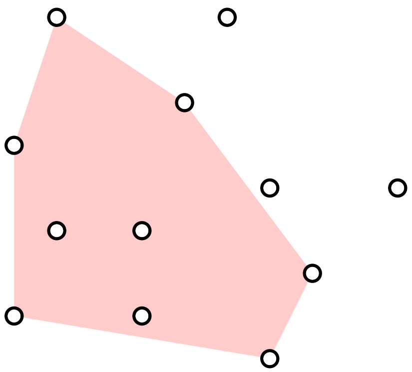

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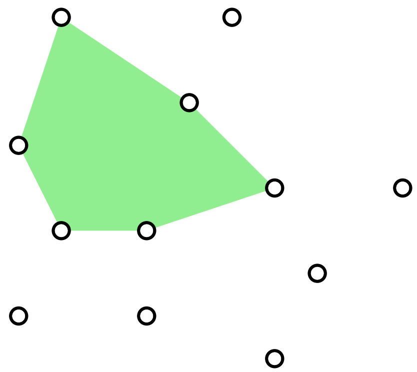

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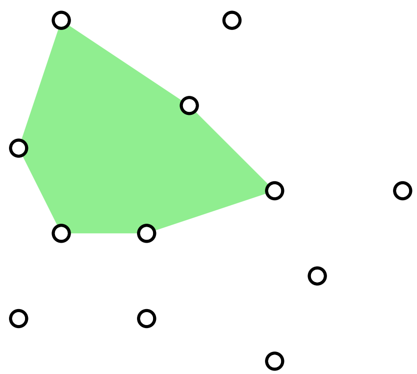

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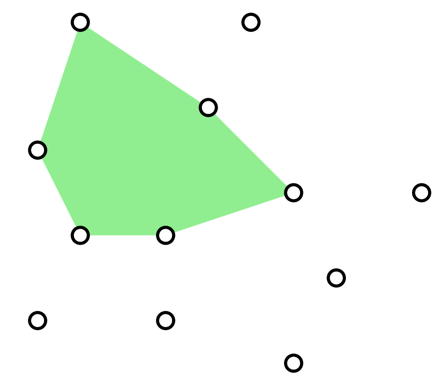

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- ▷ Testing geometric algorithms (that use only orientation predicates).

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Our "concentration" result:

There is a **vanishingly small** subset A_n of the n-point order types such that a random n-point i.i.d. sample of a square has order type in A_n **with probability** $\to \mathbf{1}$.

Random polygons

based on order types

Random polytopes 101:

How many extreme points in a random **point set** sampled **independently** from some measure μ over $\mathbb{R}^d$?

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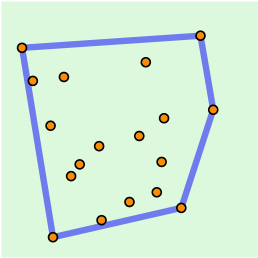

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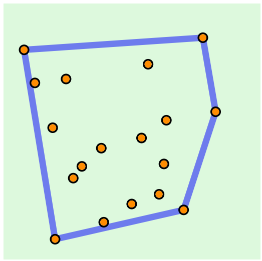

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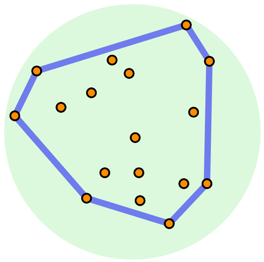

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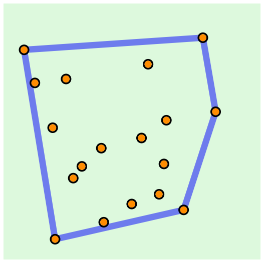

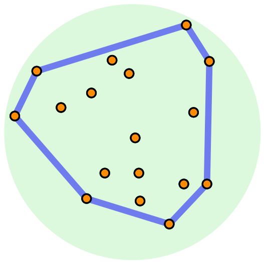

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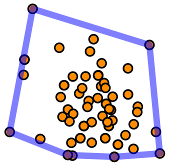

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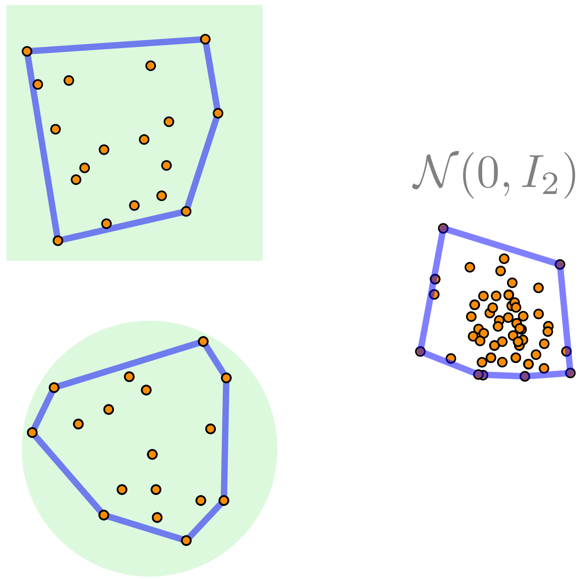

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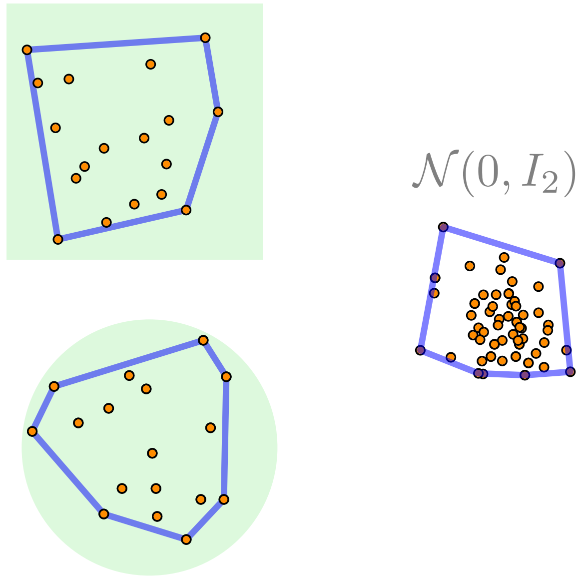

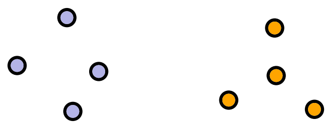

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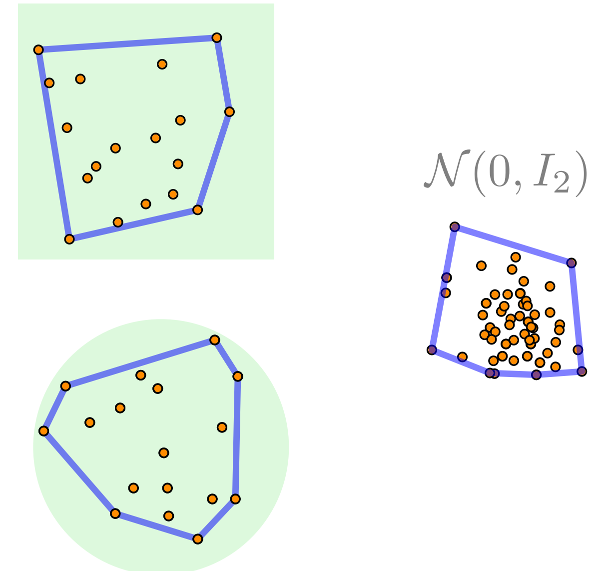

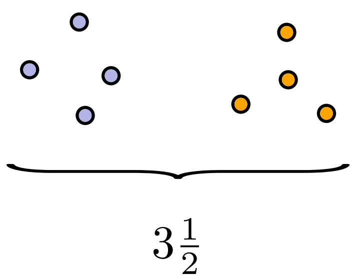

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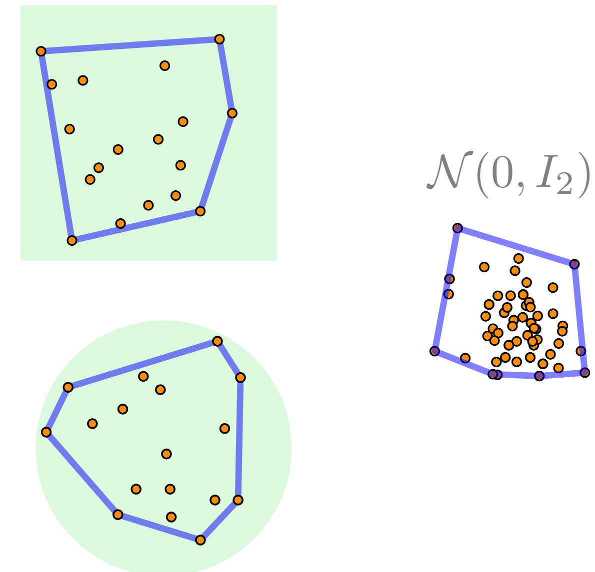

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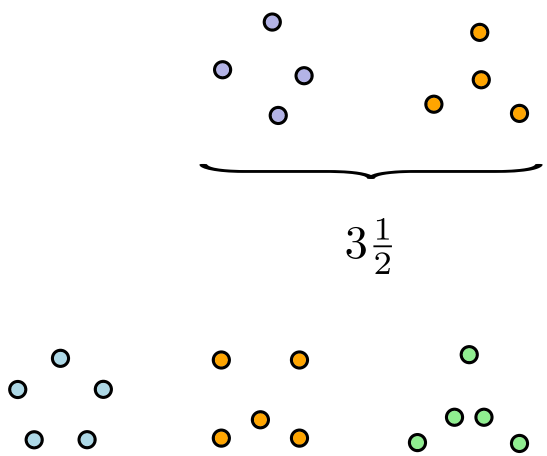

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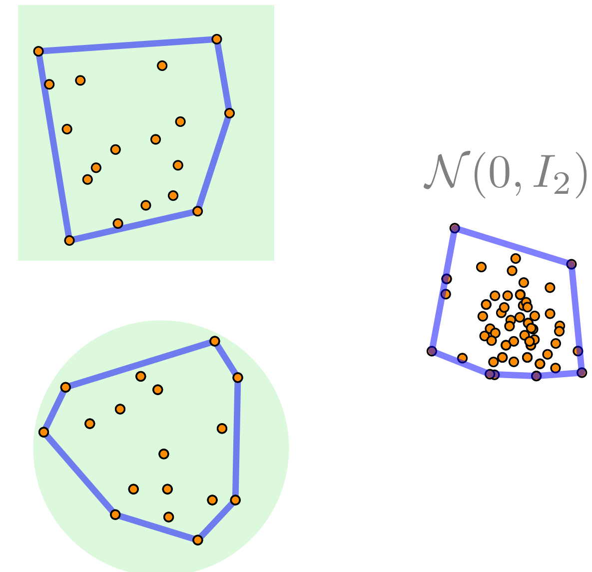

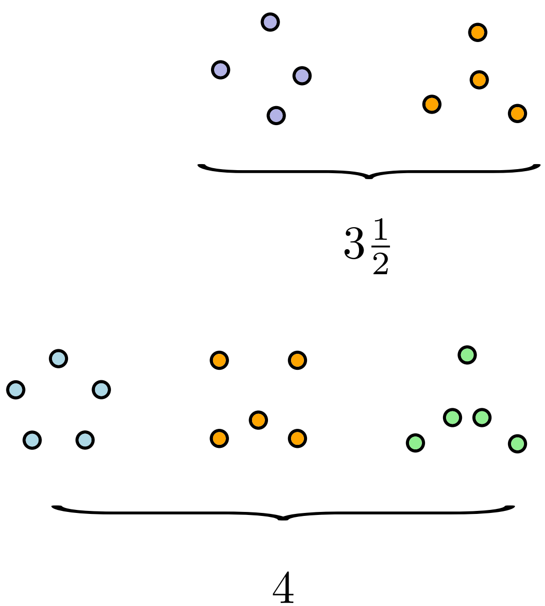

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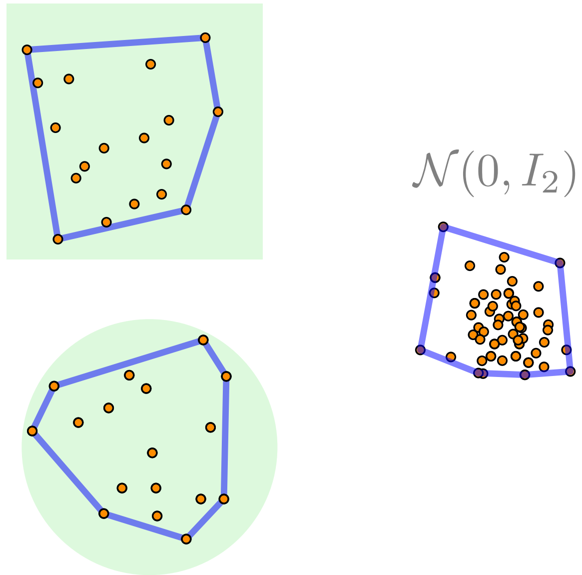

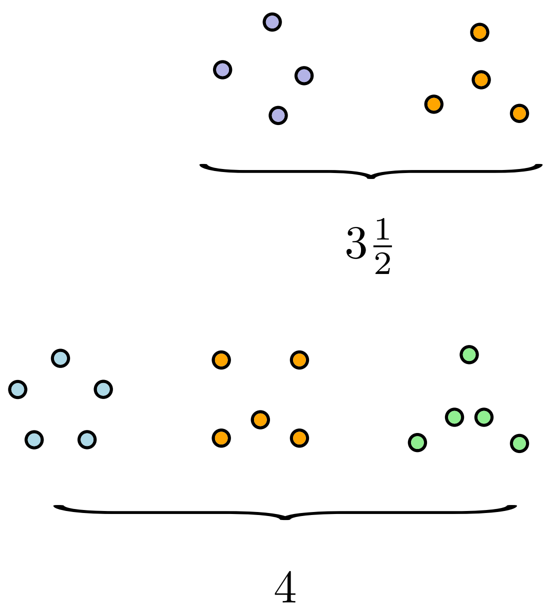

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Answer: $\sim \mathbf{4}$ as $\mathbf{n} \to \infty$.

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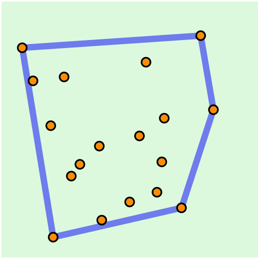

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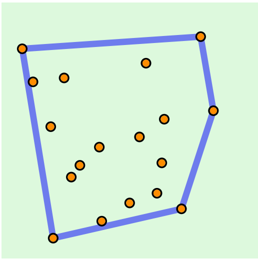

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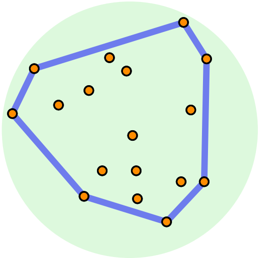

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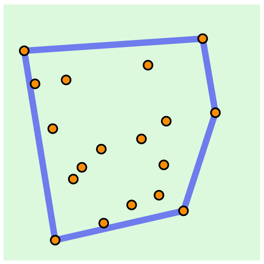

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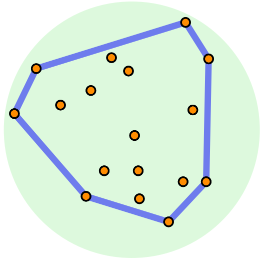

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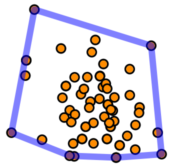

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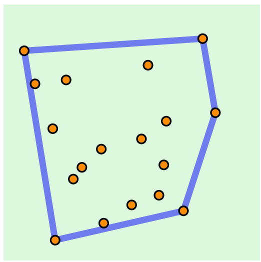

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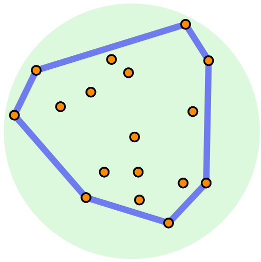

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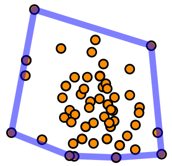

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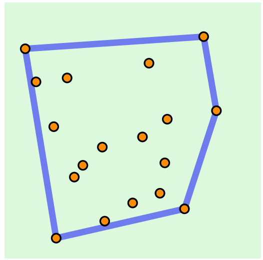

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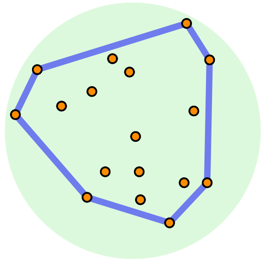

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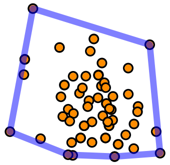

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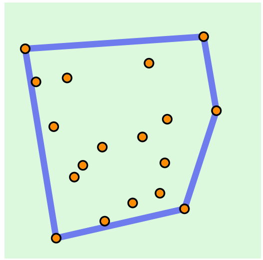

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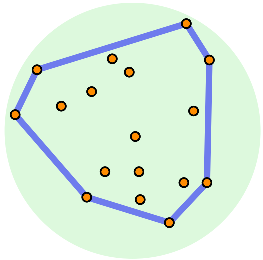

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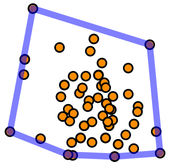
The order types of these random point sets are concentrated.



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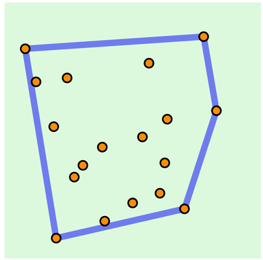

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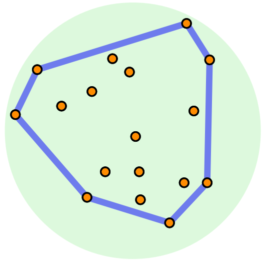

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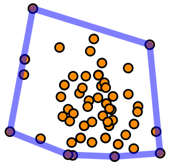

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$A_n \stackrel{\text{def}}{=}$ the order types with **at least** $e \stackrel{\text{def}}{=} \frac{\mathbb{E}[Z_n]}{2}$ extreme points.

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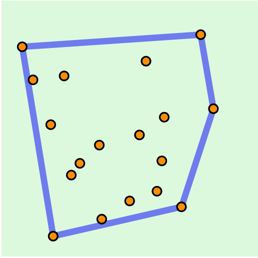

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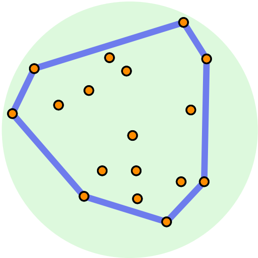

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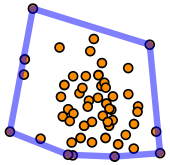

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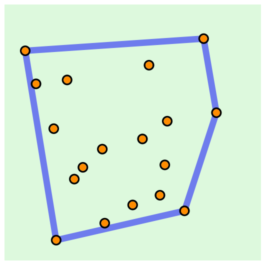
The proportion of n-point order types with $\geq e$ extreme points is $\leq \frac{4}{e}$.
(Markov's inequality applied to the equiprobable distribution.)



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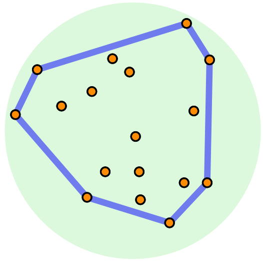
Yet, many models of **random point sets** favor sets with **many** extreme points.

$Z_n = \#$ extremes in a random n-point set.
$\mathbb{E}\left[Z_n\right] \to \infty$ and $\mathrm{Var}\left[Z_n\right] = \Theta\left(\mathbb{E}\left[Z_n\right]\right)$.
[Renyi-Sulanke][Vu][Bárány-Reitzner]



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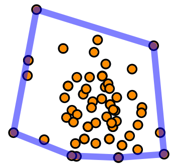

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Bienaymé-Chebyshev

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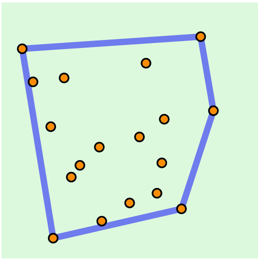

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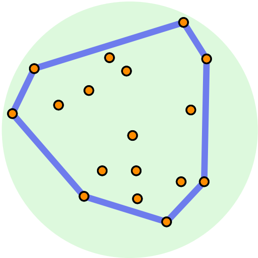

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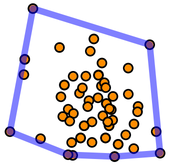

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Our main results

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Averaging on subsets

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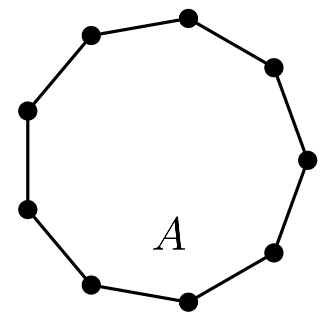

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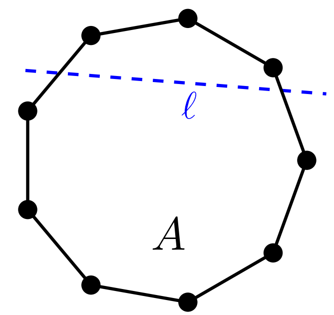

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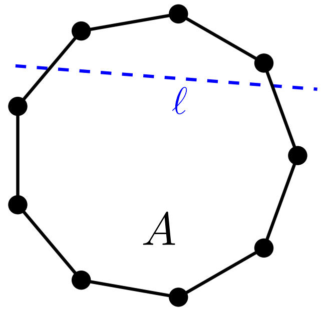

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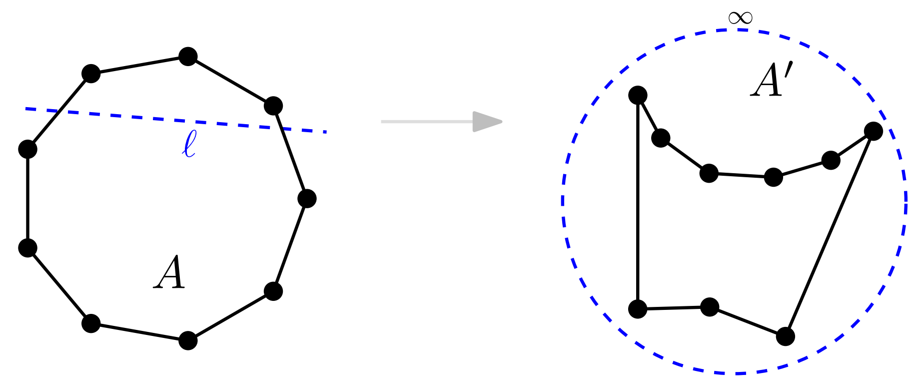

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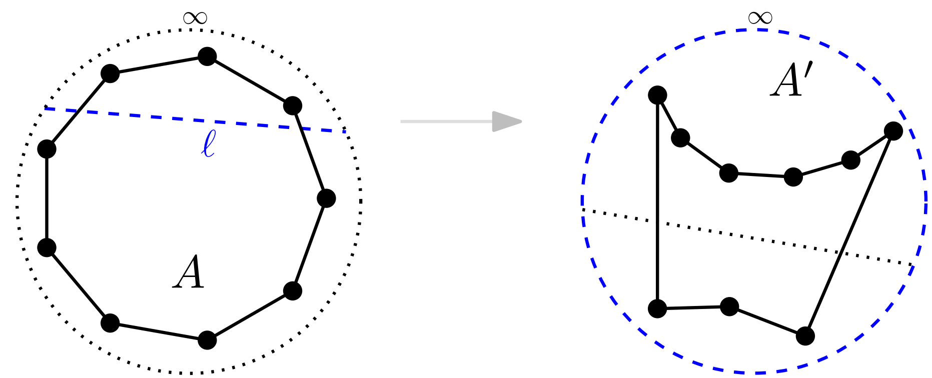

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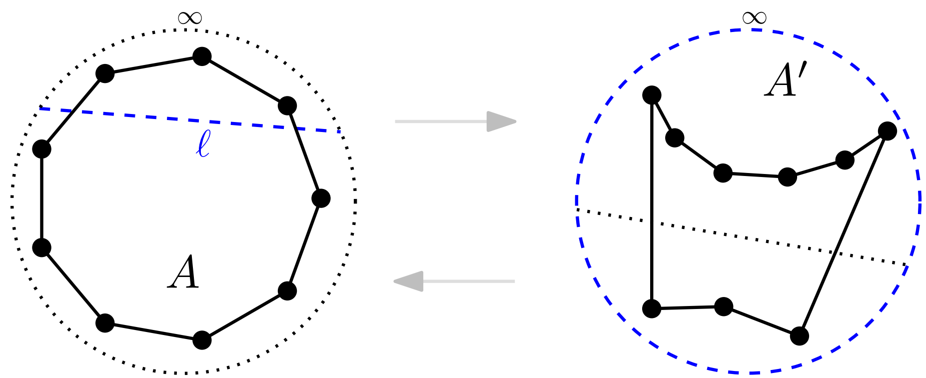

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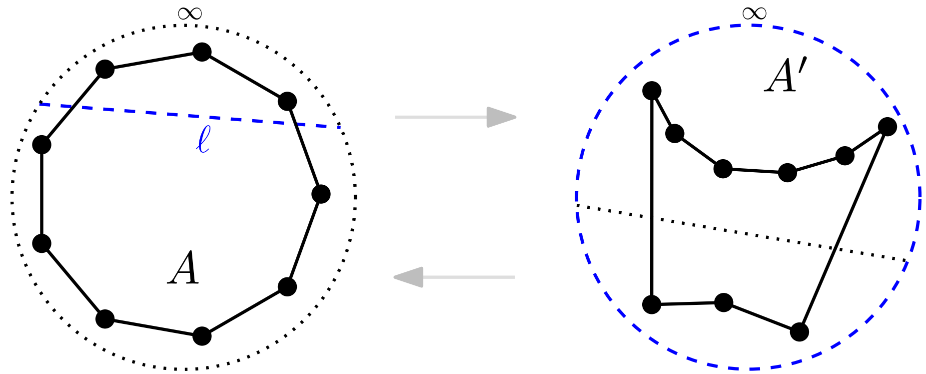

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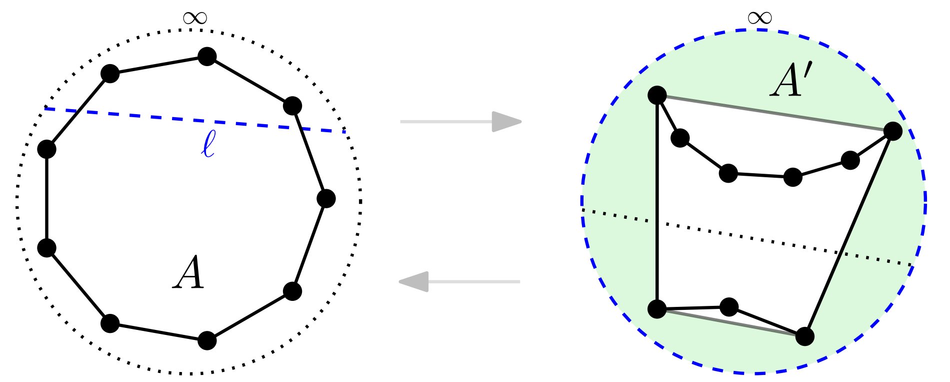

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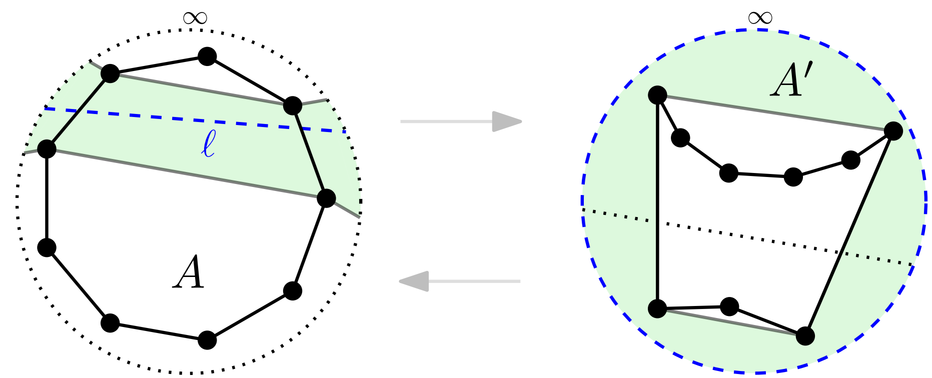

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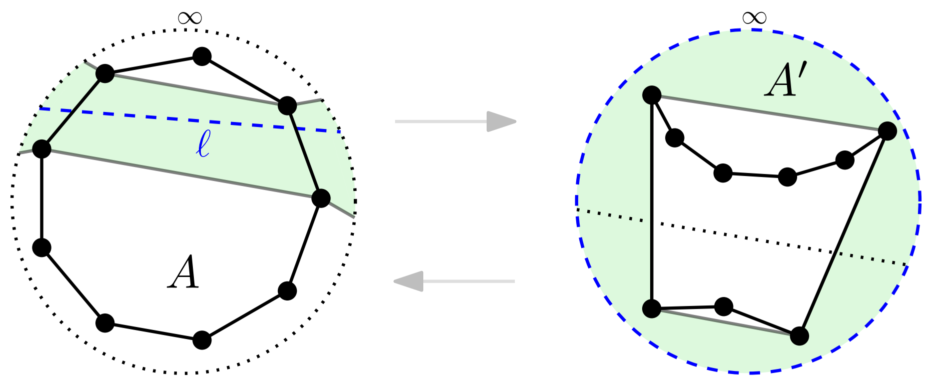

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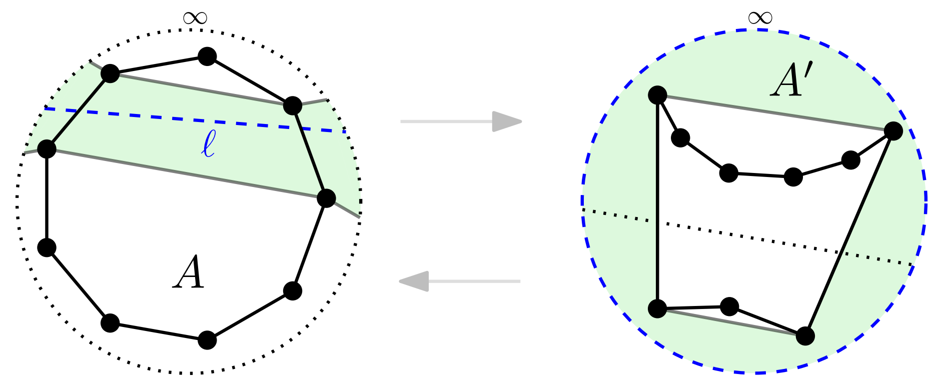

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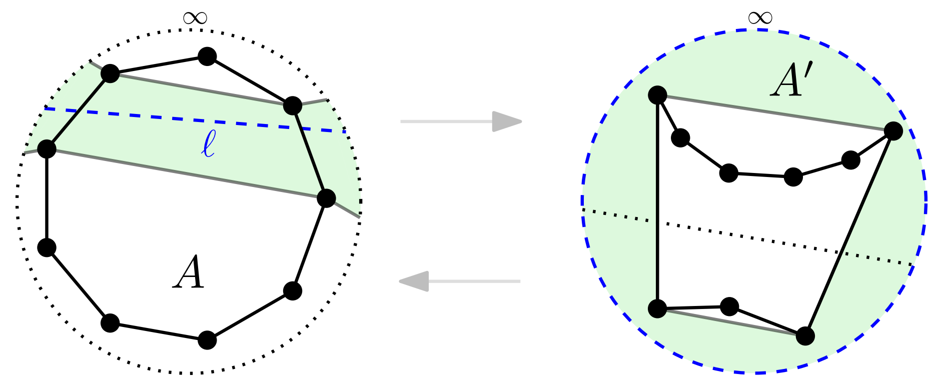

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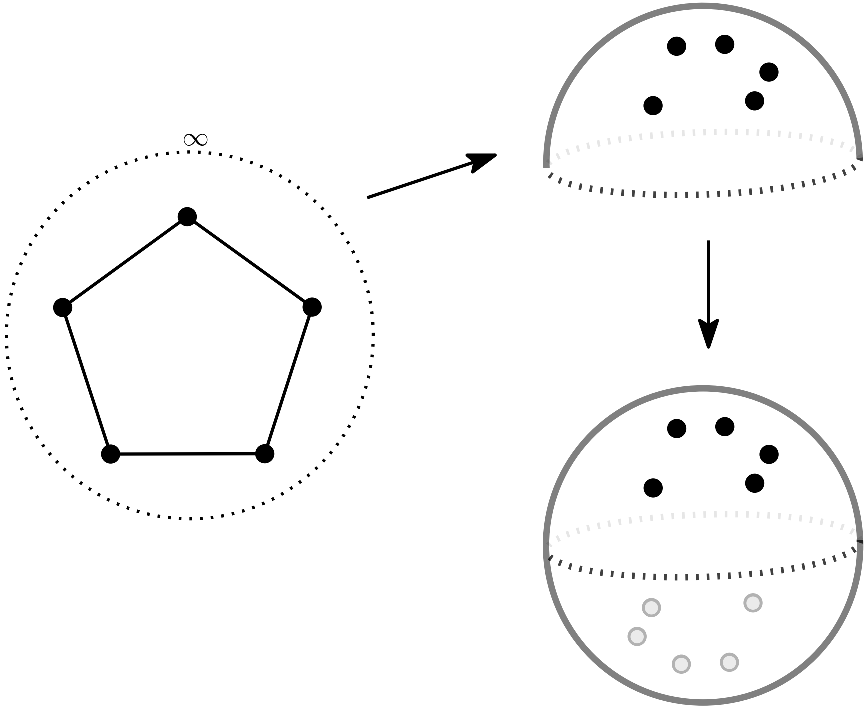
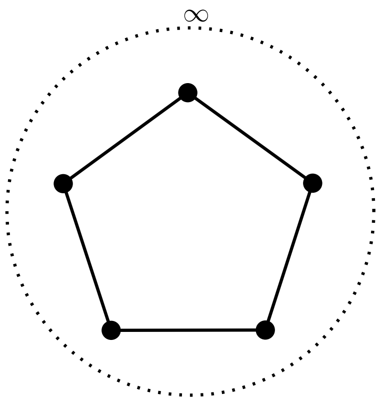
We further **divide up** order types into classes of order types equal **up to projective transforms**.

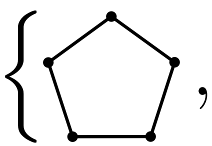
Then, we average the number of extreme points within each class.

Projective order types



What does a "class of order types equal up to projective transforms" look like?





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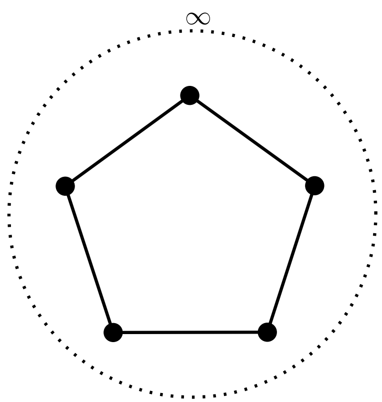

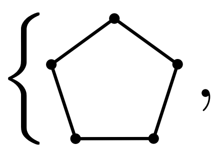

Invariance under **affine transforms**

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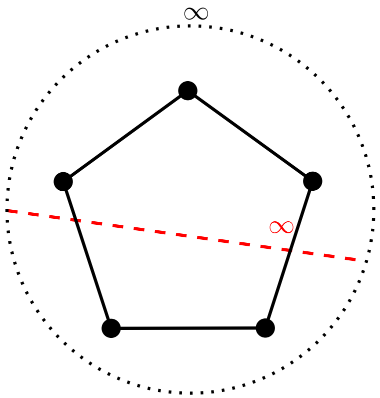

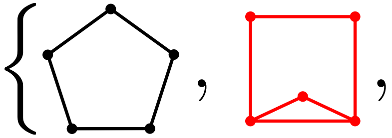

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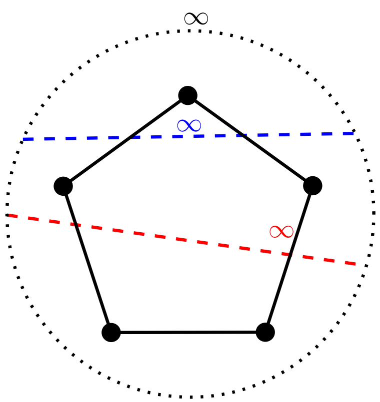

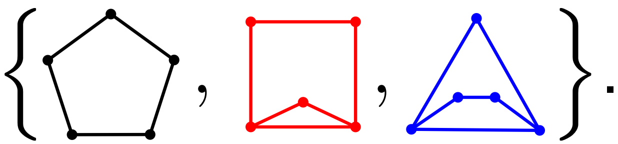

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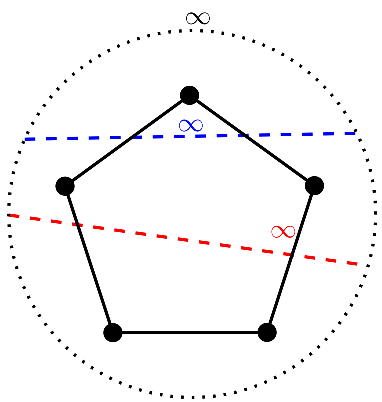
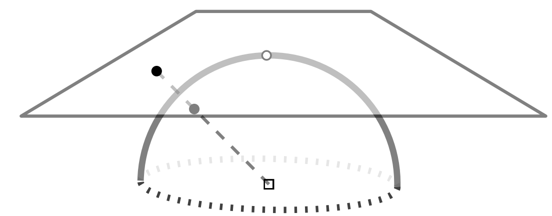
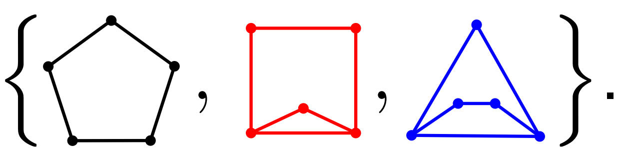
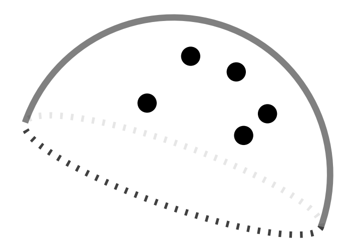




$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mapsto \frac{1}{\sqrt{x^2+y^2+1}} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$





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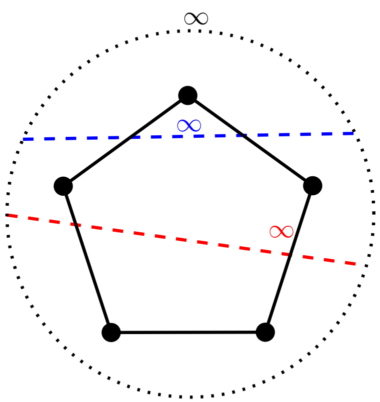

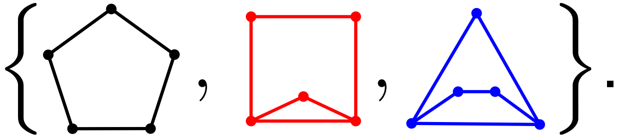

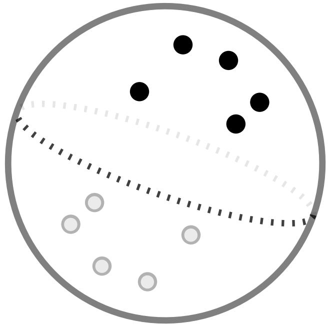

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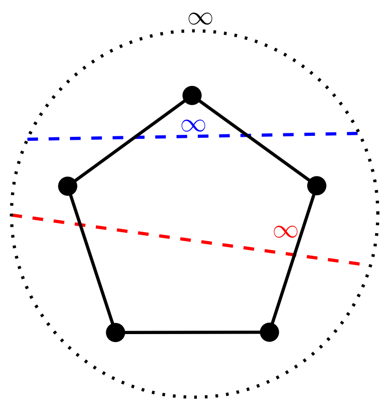

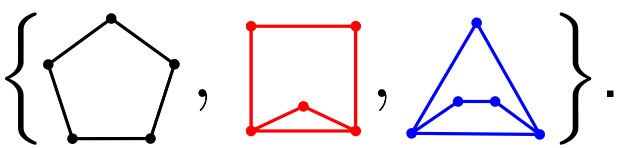

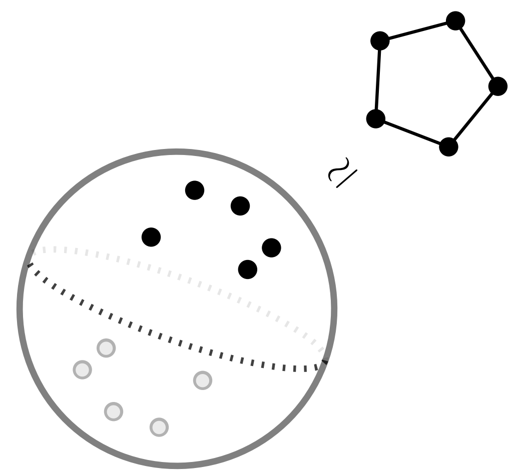

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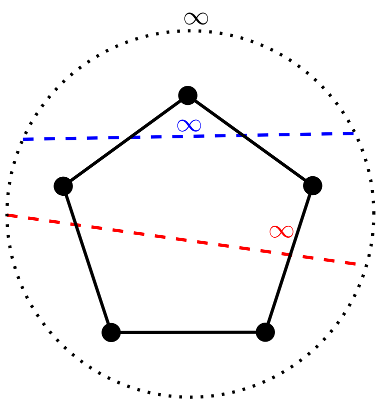

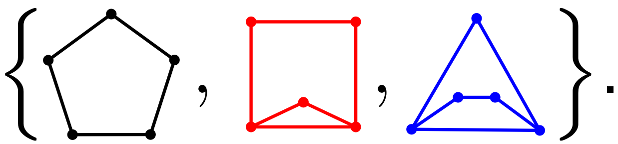

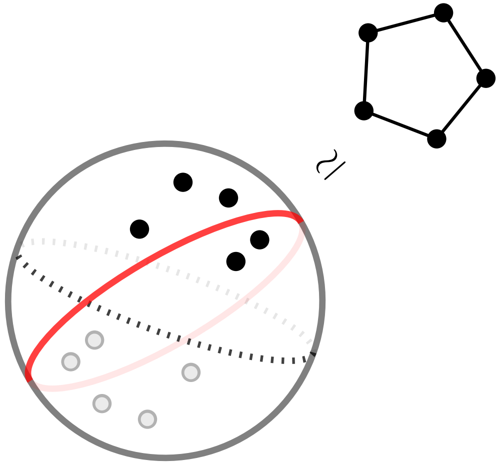

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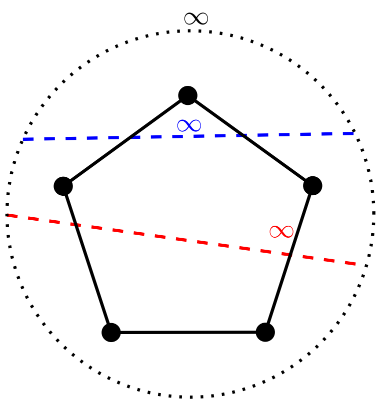

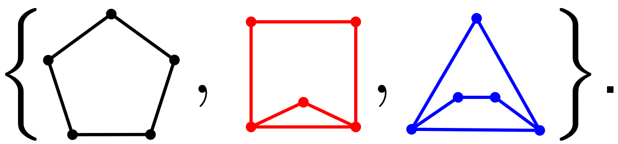

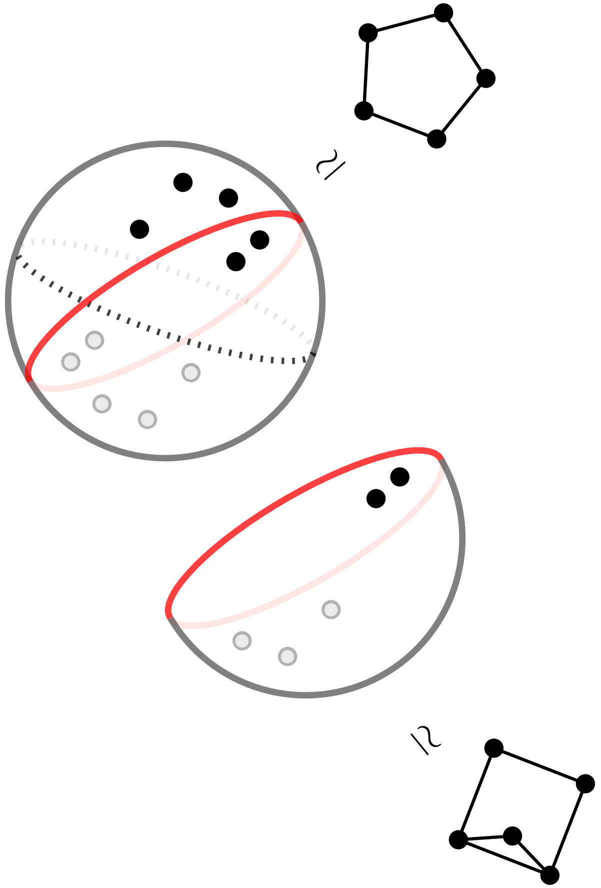

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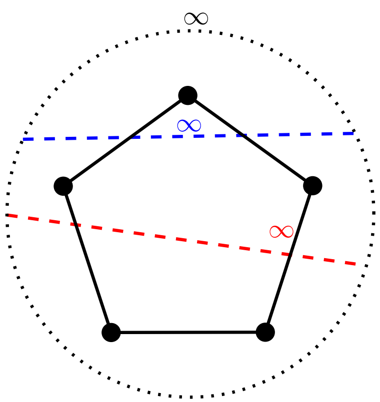

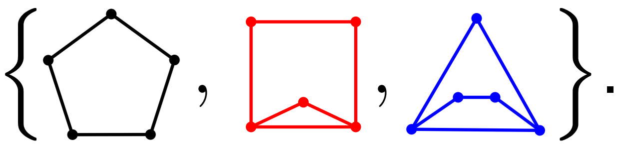

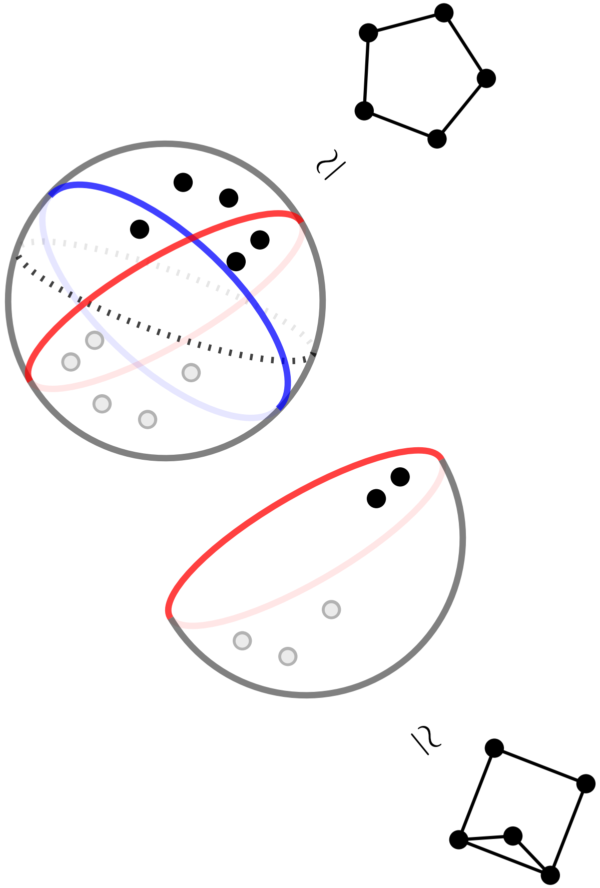

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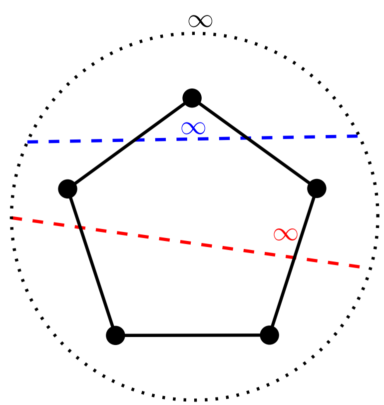

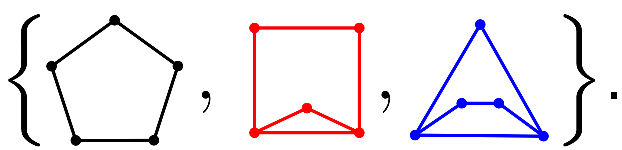

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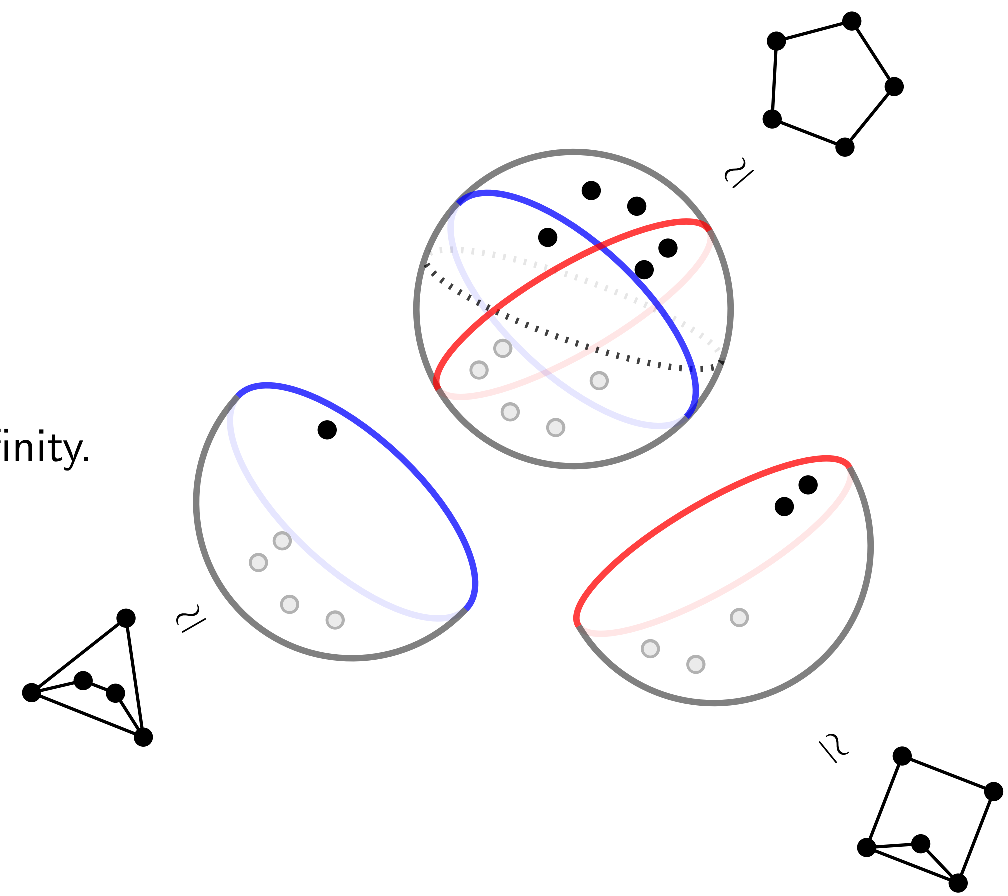

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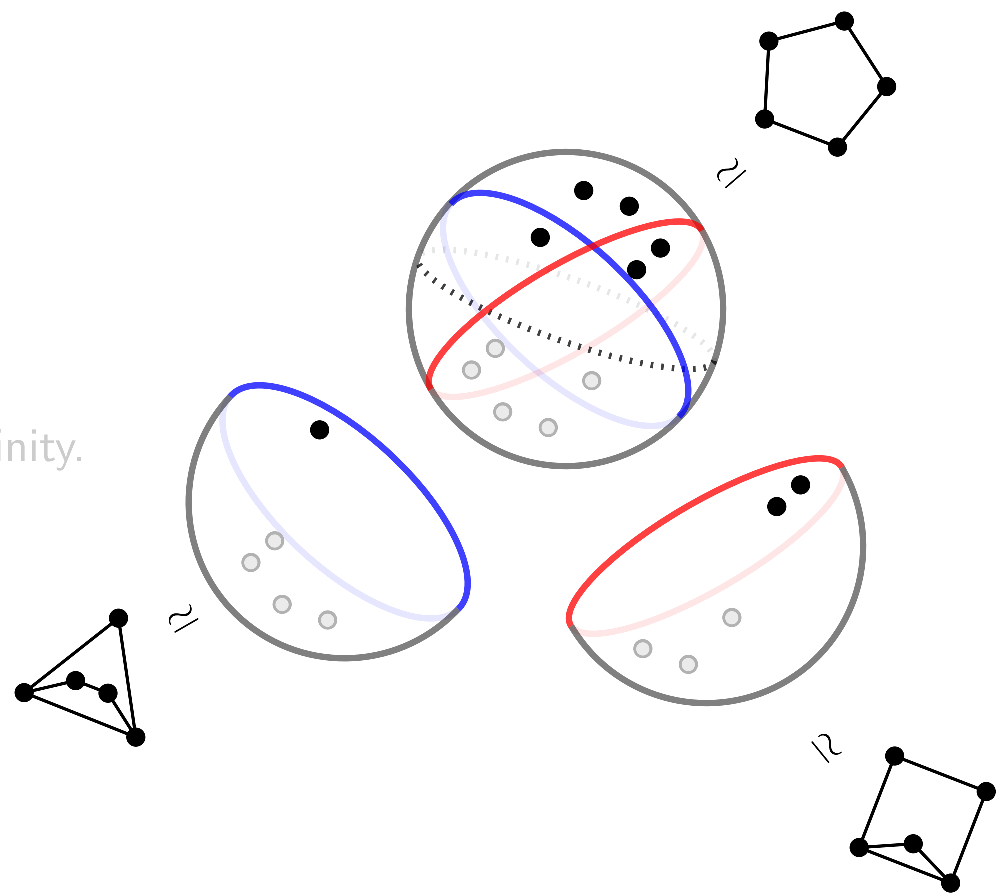
Represent a "class of order types of $\mathbb{R}^2$ equal up to projective transforms" by a "symmetric order type" on $\mathbb{S}^2$



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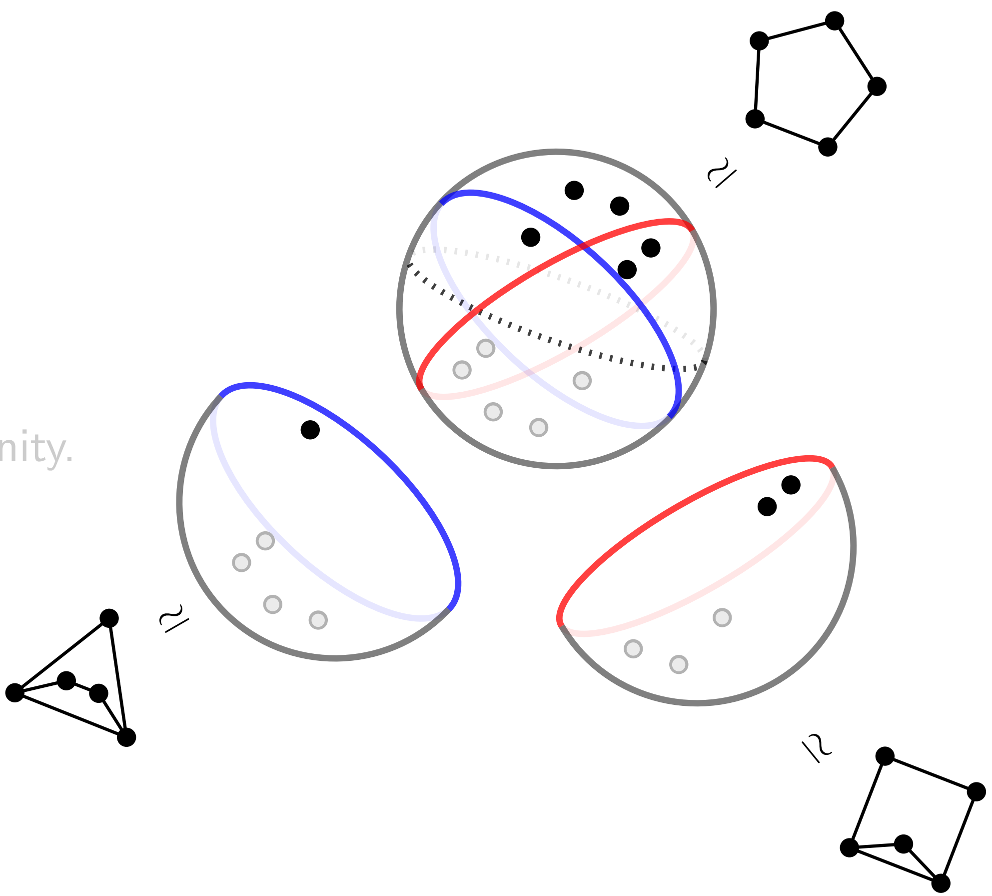

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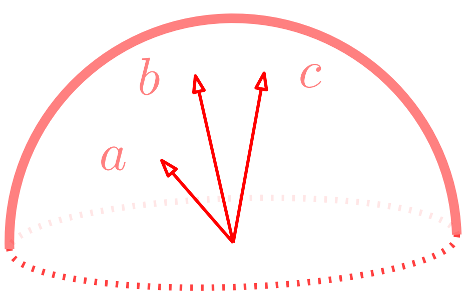
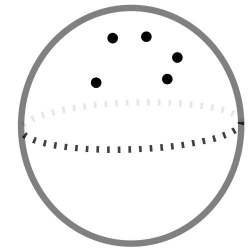


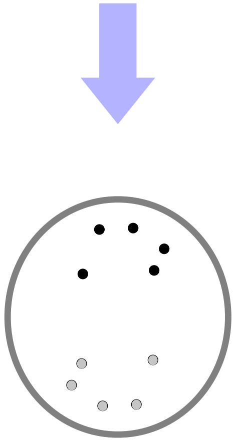
$\chi(a,b,c) \stackrel{\text{def}}{=} \text{sign}\det(a,b,c)$

An affine point set.



An affine point set.

Its "projective closure".



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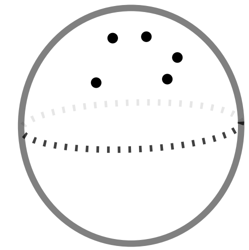

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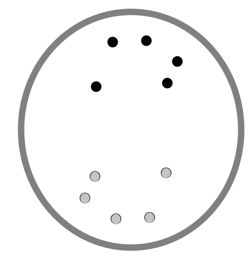

All order types of
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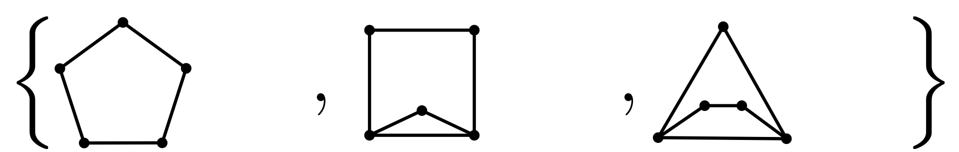

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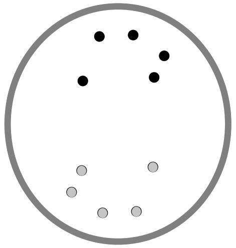

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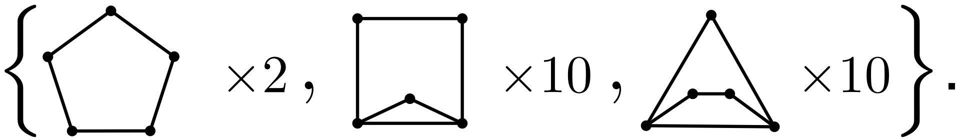
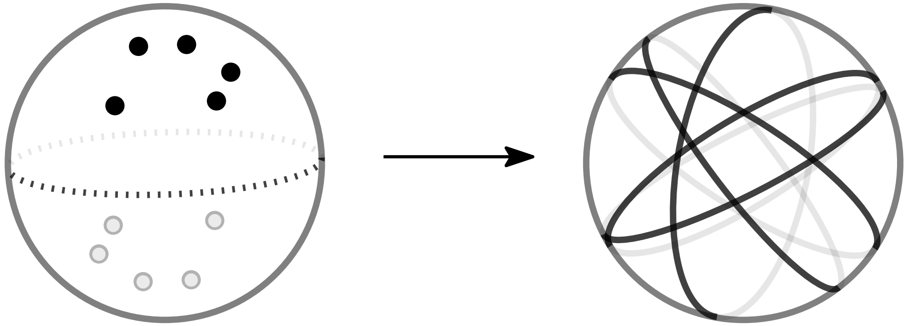
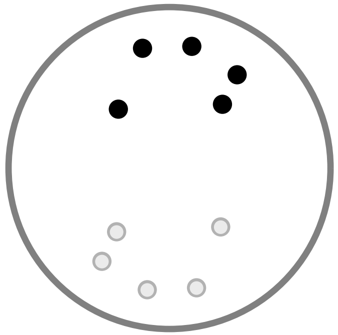


Duality



P a projective $2n$-point set.



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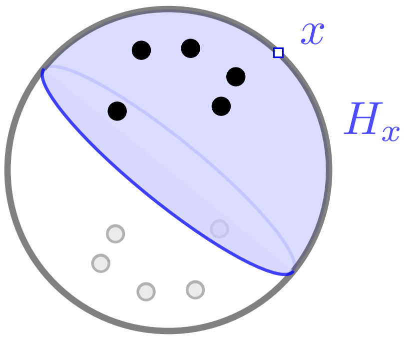

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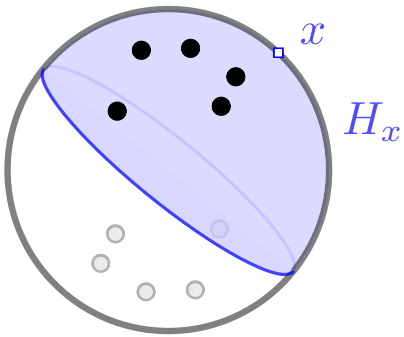

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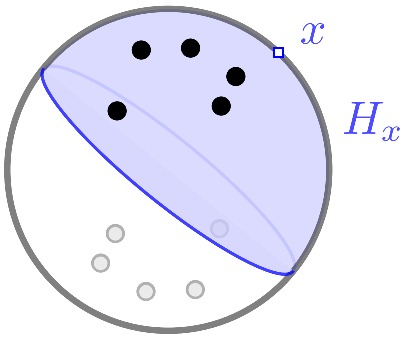

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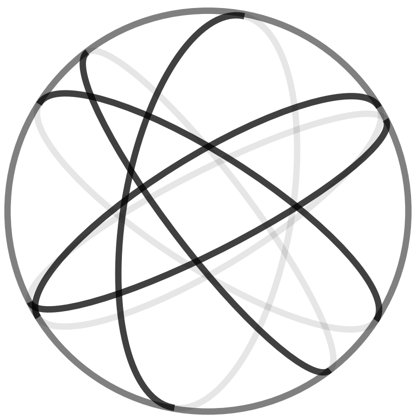

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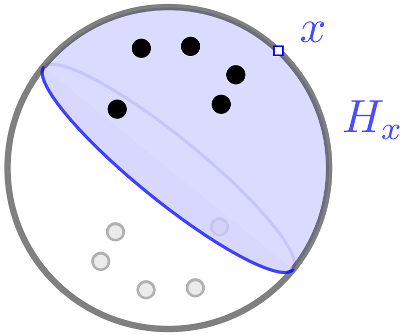

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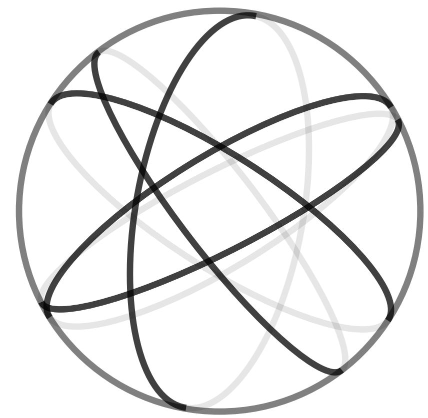

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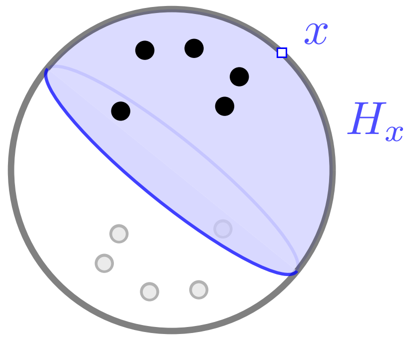

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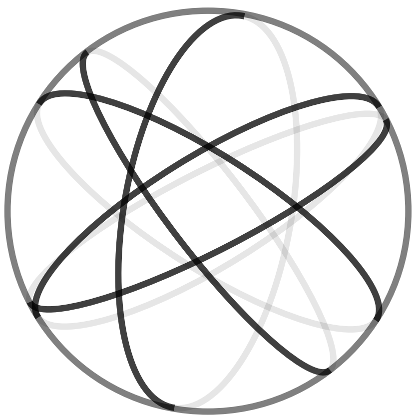

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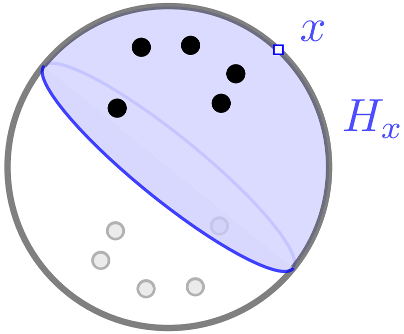

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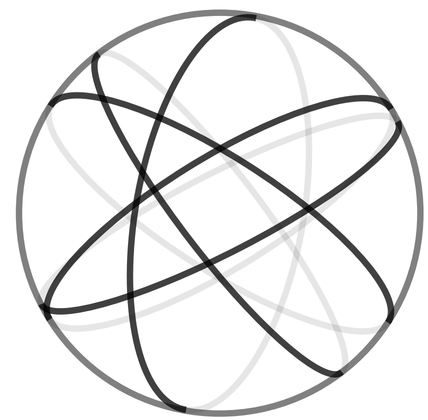

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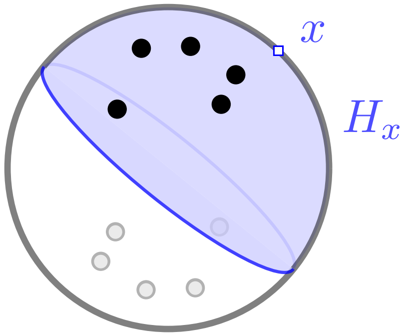

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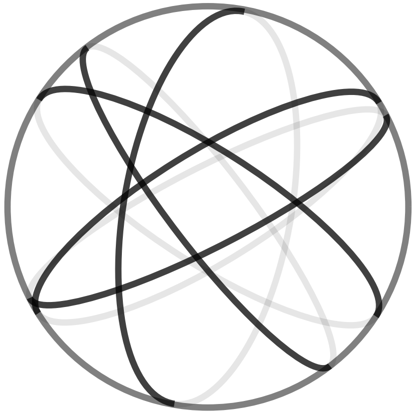

Affine hemisets of P
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$\boxed{\text{How can we control the multiplicities?}}$

Symmetries
and labeled order types

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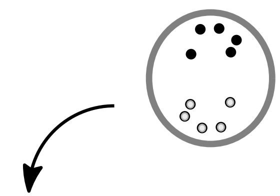

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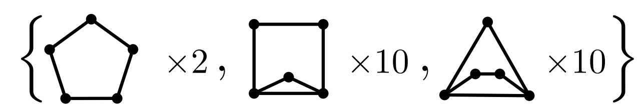

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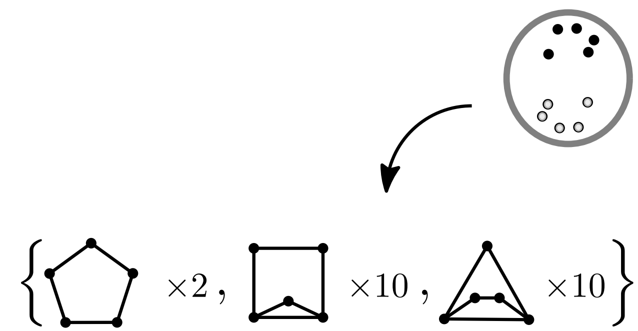

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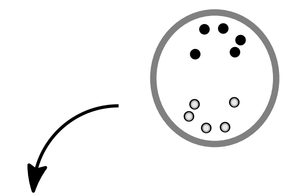

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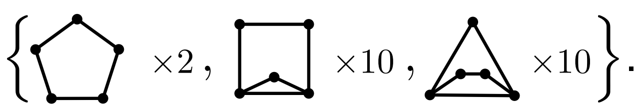

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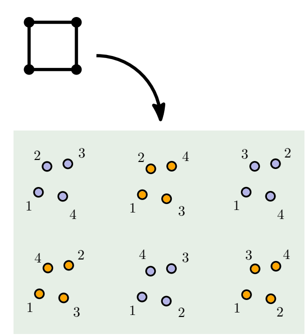

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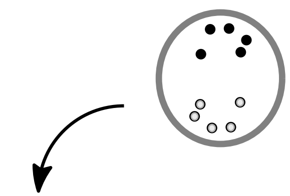

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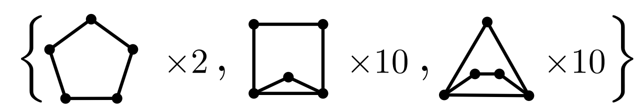

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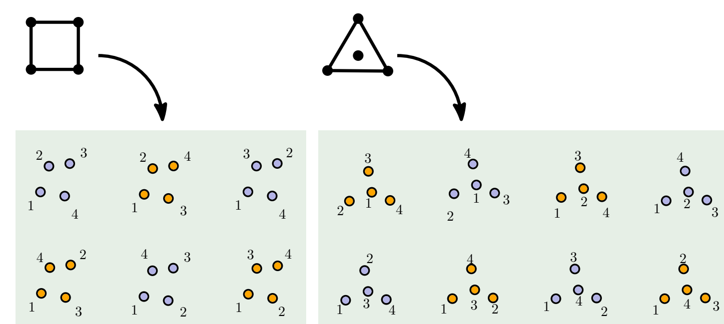

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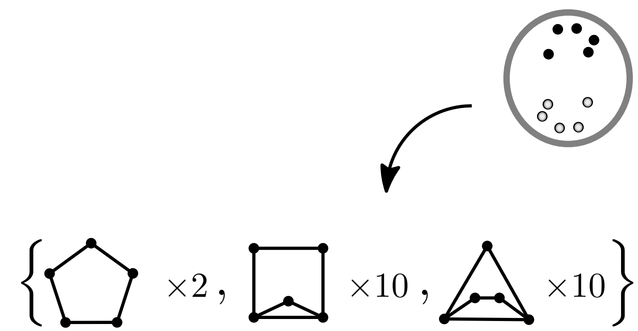

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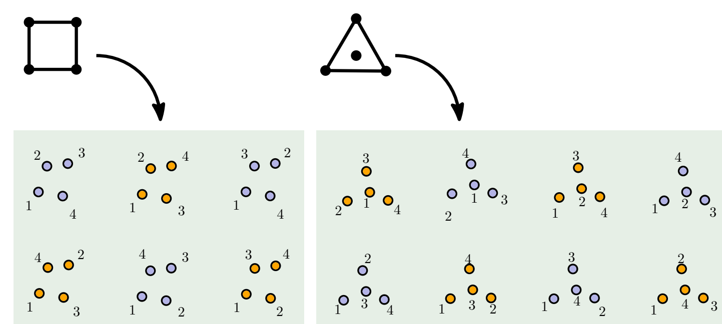
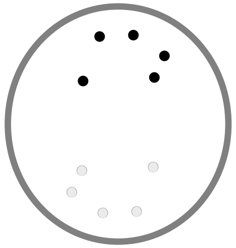
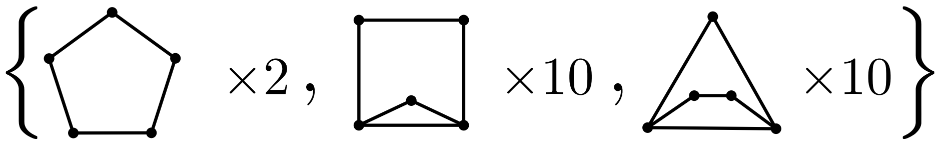


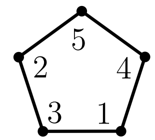
Some **unlabeled** order types may decompose into more **labeled** order types than others.

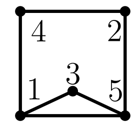
These two biases cancel each other out!



$\leftrightarrow$



Let us compare the probabilities of 

and 

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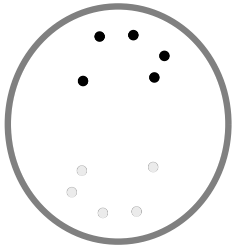
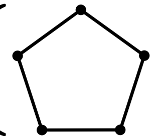
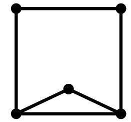
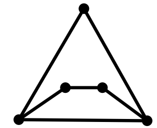
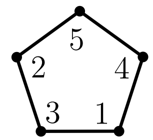
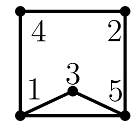
 $\leftrightarrow$ $\Big\{$  $\times 2$,  $\times 10$,  $\times 10 \Big\}$.

Let us compare the probabilities of 

and 

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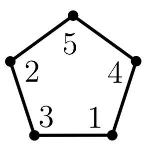
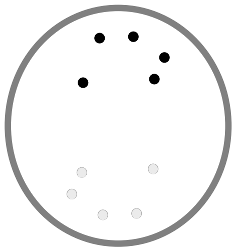
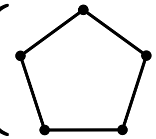
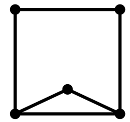
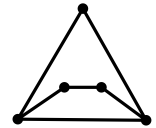
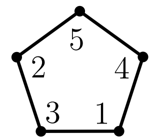
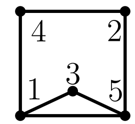
$\Pr\Big(\bar{A} \simeq$ 

$\Big) =$

 $\leftrightarrow$ $\Bigg\{$  $\times 2$,  $\times 10$,  $\times 10 \Bigg\}$.

Let us compare the probabilities of 

and 

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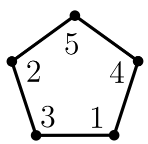
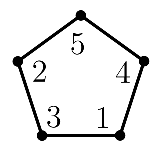
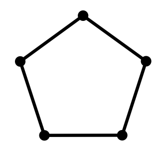
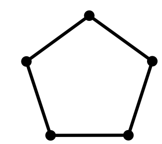
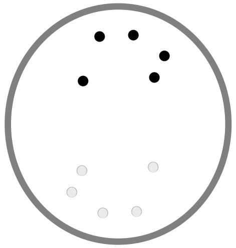
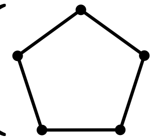
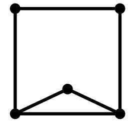
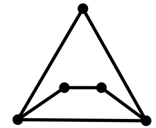
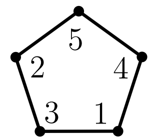
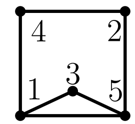
$\Pr\Big($ $\bar{A} \simeq$ 

$\Big) =$ $\Pr\Big($ $\bar{A} \simeq$ 

$\Big|\ A \simeq$  $\Big) \times \Pr\Big($ $A \simeq$  $\Big)$

 $\leftrightarrow$ $\Big\{$  $\times 2$,  $\times 10$,  $\times 10$ $\Big\}$.

Let us compare the probabilities of 

and 

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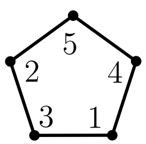
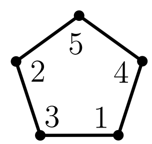
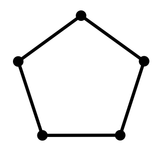
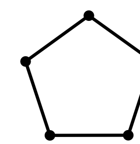
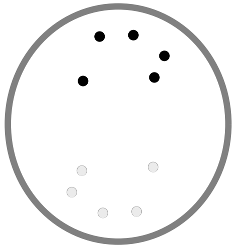
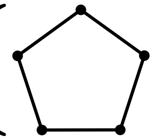
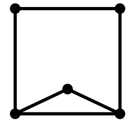
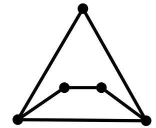
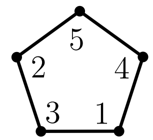
$\Pr\Big(\bar{A} \simeq$ 

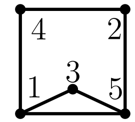
$\Big) = \Pr\Big(\bar{A} \simeq$ 

$\Big|\, A \simeq$  $\Big) \times \Pr\Big(A \simeq$  $\Big)$

$$= \frac{5}{120} \times \frac{2}{22}$$

 $\leftrightarrow$ $\Big\{$  $\times 2$,  $\times 10$,  $\times 10 \Big\}$.

Let us compare the probabilities of 

and 

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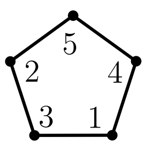
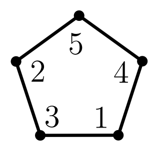
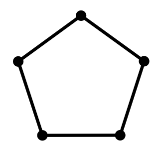
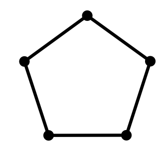
$\Pr\Big(\bar{A} \simeq$  $\Big) = \Pr\Big(\bar{A} \simeq$  $\Big| \, A \simeq$  $\Big) \times \Pr\Big(A \simeq$  $\Big)$

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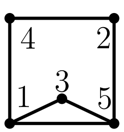
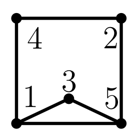
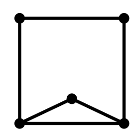
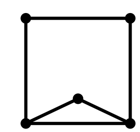
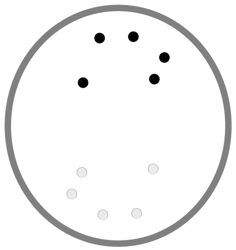
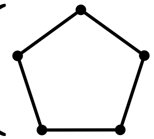
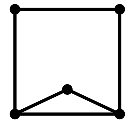
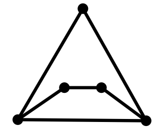
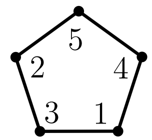
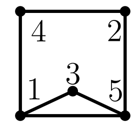
$\Pr\Big(\bar{A} \simeq$  $\Big) = \Pr\Big(\bar{A} \simeq$  $\Big| \, A \simeq$  $\Big) \times \Pr\Big(A \simeq$  $\Big)$

 $\leftrightarrow$ $\Big\{$  $\times 2$,  $\times 10$,  $\times 10 \Big\}$.

Let us compare the probabilities of  and .

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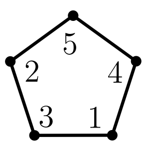
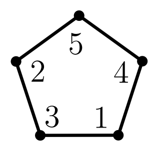
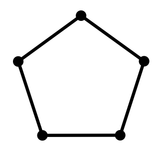
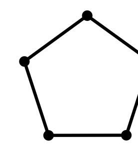
$\Pr\Big(\bar{A} \simeq$  $\Big) = \Pr\Big(\bar{A} \simeq$  $\Big|\, A \simeq$  $\Big) \times \Pr\Big(A \simeq$  $\Big)$

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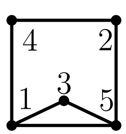
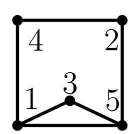
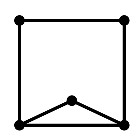
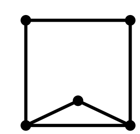
$\Pr\Big(\bar{A} \simeq$  $\Big) = \Pr\Big(\bar{A} \simeq$  $\Big|\, A \simeq$  $\Big) \times \Pr\Big(A \simeq$  $\Big)$

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Symmetry of $S \subset \mathbb{S}^2 \stackrel{\mathsf{def}}{=}$ orientation-preserving bijection $S \to S$.

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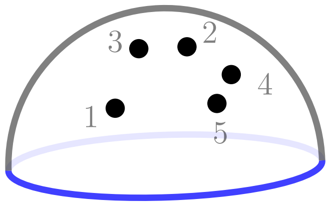

Cyclic group $\mathbb{Z}_5$

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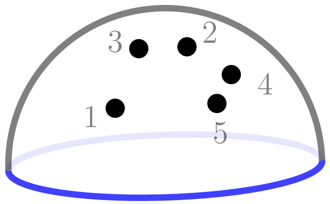

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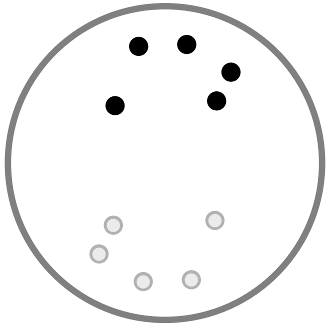

Dihedral group D_5

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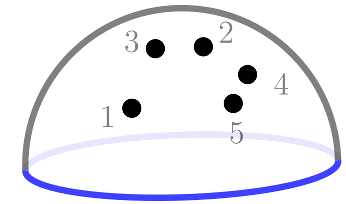
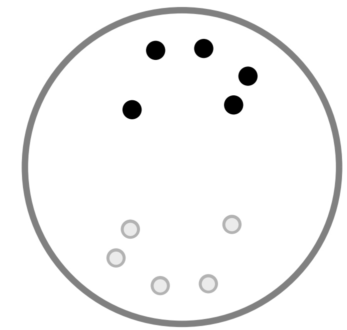


Cyclic group $\mathbb{Z}_5$

The symmetries of a projective sets **act on** its affine subsets.



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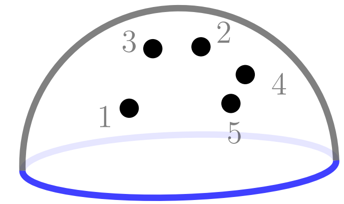

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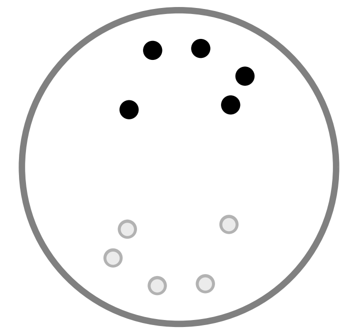

Dihedral group D_5

P a projective set, G its symmetry group, A an affine subset of P.

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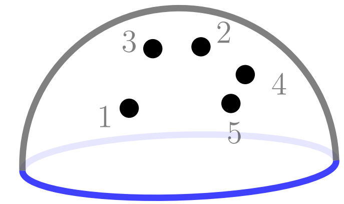

Cyclic group $\mathbb{Z}_5$

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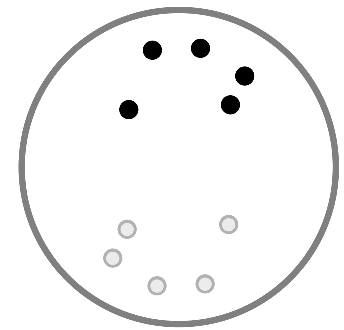

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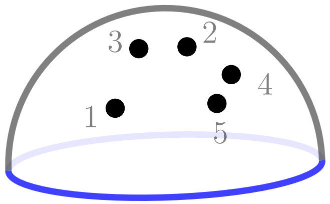

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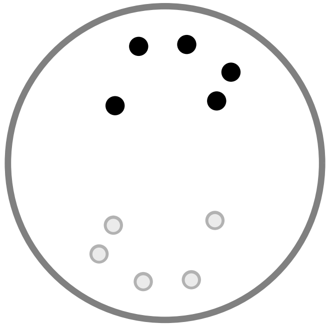

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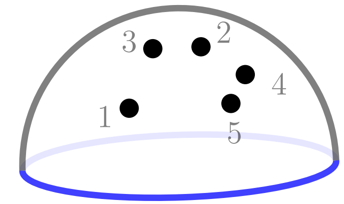

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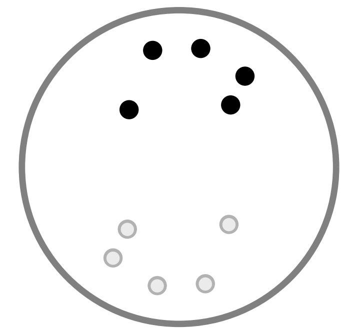

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1st source of bias

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Altogether...

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Time to wrap-up!

Conjecture. Order types of n i.i.d. points whose number of extreme points goes to ∞ as $n \to \infty$ exhibit concentration.

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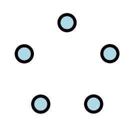
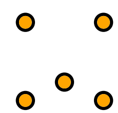
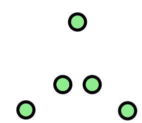
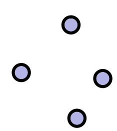
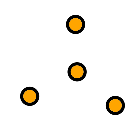
Sample uniformly in    then drop a random point

yields  with proba 0.6, and  with proba 0.4

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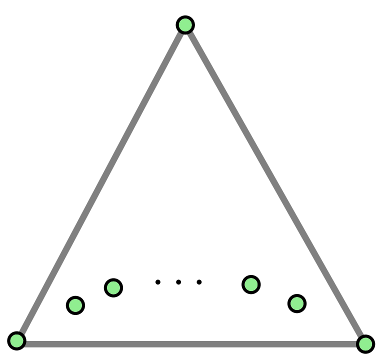

Thank you for your attention!

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There are $\frac{|\mathsf{Sym}(P)|}{|\mathsf{Sym}(A)|}$ affine hemisets with same order type as A.

$n \stackrel{\text{def}}{=} |A|$ and $\mathsf{Sym}(X) \stackrel{\text{def}}{=}$ the symmetry group of X.

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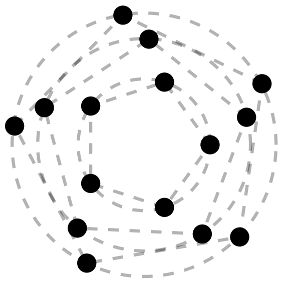

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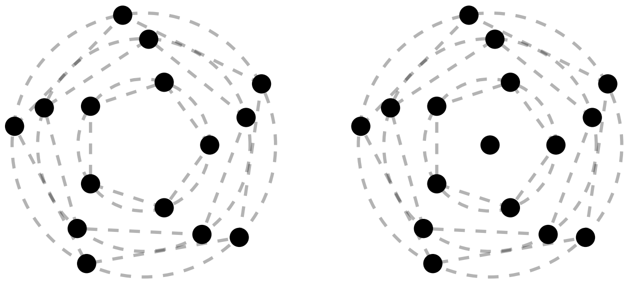

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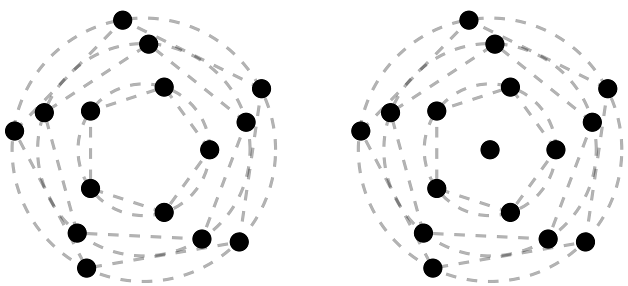

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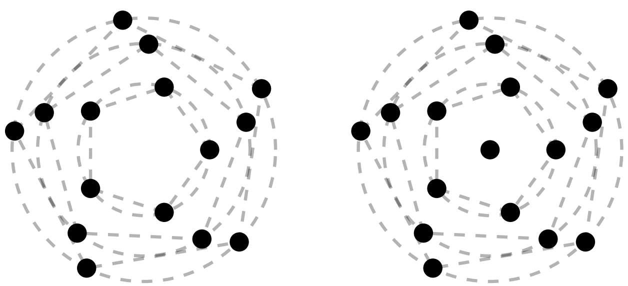

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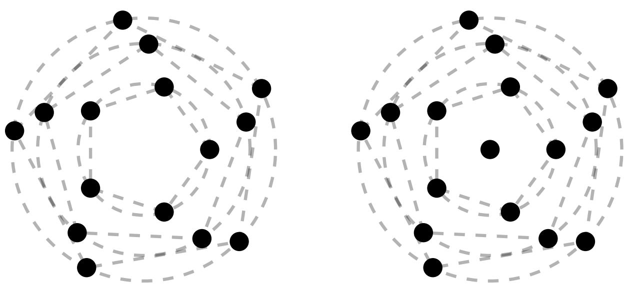

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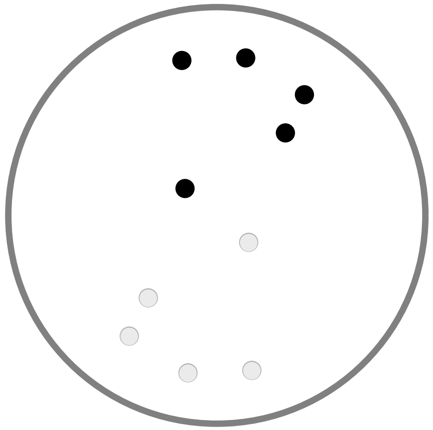

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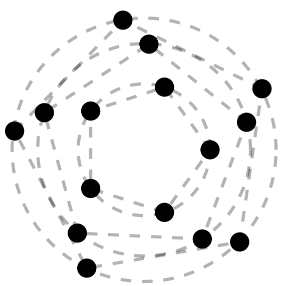

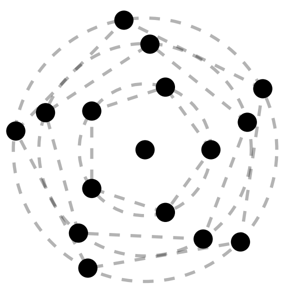

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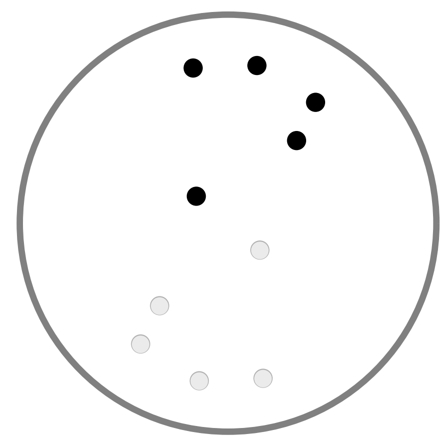

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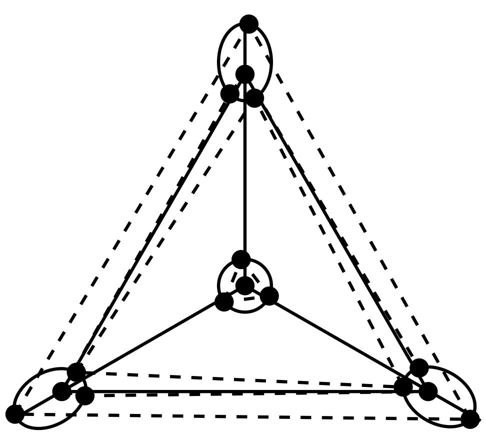

Each of these groups occurs
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